Stoke's Theorem :

Statement :

* If $\vec{F}$ is any continous differentiable vector function and S is the surface enclosed by a curve C then single intregral

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, ds.$$

* where, $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

Example : 1

Verify stoke's theorem for a function $\vec{F} = (y - z + 2)\vec{i} + (yz + 4)\vec{j} - 4z\vec{k}$ where S is the open surface of the cube $x = 0$, $x = 2$

$y = 0$, $y = 2$, $z = 2$ about the xy plane

Soln: Given,

$$\vec{F} = (y-z+2)\vec{i} + (yz+4)\vec{j} - 4z\vec{k}.$$

Formula: $\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\cdot ds.$

RHS:

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y-z+2 & yz+4 & -4z \end{vmatrix}$$

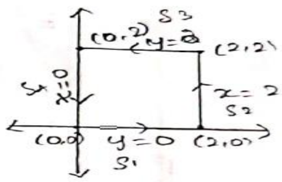
$$= \vec{i}\left[\frac{\partial}{\partial y}(-4z) - \frac{\partial}{\partial z}(yz+4)\right] - \vec{j}\left[\frac{\partial}{\partial x}(-4z) - \frac{\partial}{\partial z}(y-z+2)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(yz+4) - \frac{\partial}{\partial y}(y-z+2)\right]$$

$$= \vec{i}(-y) - \vec{j}(1) + \vec{k}(-1)$$

$$= -y\vec{i} - \vec{j} - \vec{k}$$

$$= y\vec{i} + \vec{j} + \vec{k}$$



surface	equation	$\hat{n}$	ds
S_1	$x=0$	$-\vec{i}$	dydz
S_2	$x=2$	$\vec{i}$	dydz
S_3	$y=0$	$-\vec{j}$	dxdz
S_4	$y=2$	$\vec{j}$	dx dz
S_5	$z=2$	$\vec{k}$	dxdy

Along OA $y=0$.

$$= -\iint_{OA} (y\vec{i}+\vec{j}+\vec{k})\cdot\vec{j}\, dx\, dz$$

$$= -\int_0^2\int_0^2 dx\, dz$$

$$= -\int_0^2 (x)_0^2\, dz$$

$$= -\int_0^2 (2)dz = -\left[2(z)_0^2\right]$$

$$= -2(2) = -4$$

Along AB, $x=2$ $\hat{n} = \vec{i}$ $ds = dydz$

$$= \iint_{AB} (y\vec{i} + \vec{j} + \vec{k})\cdot\vec{i}\, dydz.$$

$$= \int_0^2 \int_0^2 (y)\, dy\, dz$$

$$= \int_0^2 \left(\frac{y^2}{2}\right)_0^2 dz = \int_0^2 \left(\frac{4}{2}\right) dz$$

$$= 2\int_0^2 dz = 2\,(z)_0^2$$

$$= 2(2) = 4.$$

Along BC: $y = 2$, $\hat{n} = \vec{j}$, $ds = dx\,dz$

$$= \iint_{BC} (y\vec{i} + \vec{j} + \vec{k}) \cdot \vec{j}\; dx\, dz$$

$$= \int_0^2 \int_0^2 dx\, dz = \int_0^2 (x)_0^2\, dz$$

$$= 2\int_0^2 dz = 2\,(z)_0^2$$

$$= 4.$$

Along OC: $x = 0$, $\hat{n} = -\vec{i}$, $ds = dy\,dz$

$$= -\iint_{OC} (y\vec{i} + \vec{j} + \vec{k}) \cdot \vec{i}\; dy\, dz$$

$$= -\int_0^2 \int_0^2 y\, dy\, dz$$

$$= -\int_0^2 \left(\frac{y^2}{2}\right)_0^2 dz = -\int_0^2 \frac{4}{2}\, dz$$

$$= -2\,(z)_0^2 = -4.$$

Along $z = 2$, $\hat{n} = \vec{k}$, $ds = dx\,dy$

$$= \iint_{S_5} (y\vec{i} + \vec{j} + \vec{k}) \cdot \vec{k}\; dx\, dy$$

$$= \int_0^2 \int_0^2 dx\, dy = \int_0^2 (x)_0^2\, dy$$

$$= 2\int_0^2 dy = 2\,(y)_0^2$$

$$= 4.$$

$$\therefore S_1 + S_2 + S_3 + S_4 + S_5 = -4+4+4-4+4$$

$$RHS. = 4.$$

LHS:

$$\vec{F}\cdot d\vec{r} = [(y-z+2)\vec{i} + (yz+4)\vec{j} - 4xz\vec{k}]\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k}).$$

$$= (y-z+2)\,dx + (yz+4)\,dy - 4xz\,dz$$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_{S_1} + \int_{S_2} + \int_{S_3} + \int_{S_4} + \int_{S_5} \vec{F}\cdot d\vec{r}$$

On S_4 $x=0$, $dx=0$, $z=0$, $dz=0$.

$$\int_{BCO} \vec{F}\cdot d\vec{r} = \int_{S_1} (yz+2)\,dx + (yz+4)\,dy - 4xz\,dz$$

$$= \int_0^2 (4)\,dy = 4\int_0^2 dy$$

$$= 4(y)_0^2 = 8.$$

On OA $y=0$ $z=0$ $dy=0$ $dz=0$.

$$\int_{OA} 2\,dx = \int_0^2 2x = 2(x)_2^0 = -4.$$

On AB $x=2$, $z=0$, $dx=0$, $dz=0$.

$$\int_{AB} 4\,dy \Rightarrow \int_0 4\,dy \Rightarrow [4y]_0^2$$

On BC $y=2$ $z=0$ $dy=0$ $dz=0$

$$\int_{BC} 4\,dx \Rightarrow \int_0^2 4\,dx = [4x]_2^0 = -8$$

$$\oint_C \vec{F}\cdot d\vec{r} = 8+8-4-8 = 4.$$