



SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Coimbatore-641035.



UNIT-I Vector calculus

Stokes theorem

Stoke's Theorem

Statement:

If \vec{F} is any continuous differentiable vector function and S is the surface enclosed by a curve C then single integral $\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} ds$.

$$\text{where, } d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

Example: 1

Verify Stoke's theorem for a function $\vec{F} = (y - z + 2)\vec{i} + (y^2 + 4)\vec{j} - 4z\vec{k}$ where S is the open surface of the cube $x=0$ to $x=2$

$y=0, y=2, z=2$ about the xy plane



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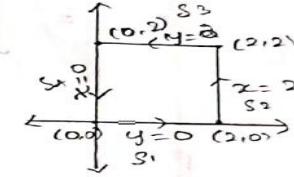
Stokes theorem

Soln: Given, $\vec{F} = (y-z+2)\vec{i} + (yz+4)\vec{j} - 4z\vec{k}$.

Formula: $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \cdot ds$.

RHS:

$$\begin{aligned}\nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z+2 & yz+4 & -4z \end{vmatrix} \\ &= \vec{i} \left[\frac{\partial}{\partial y}(-4z) - \frac{\partial}{\partial z}(yz+4) \right] - \vec{j} \left[\frac{\partial}{\partial x}(-4z) - \frac{\partial}{\partial z}(y-z+2) \right] \\ &\quad + \vec{k} \left[\frac{\partial}{\partial x}(yz+4) - \frac{\partial}{\partial y}(y-z+2) \right] \\ &= \vec{i}(-4) - \vec{j}(1) + \vec{k}(-1) \\ &= -y\vec{i} - \vec{j} + \vec{k} \\ &= y\vec{i} + \vec{j} + \vec{k}\end{aligned}$$



surface	equation	\hat{n}	ds
S_1	$x=0$	$-\vec{i}$	$dydz$
S_2	$x=2$	\vec{i}	$dydz$
S_3	$y=0$	$-\vec{j}$	$dxdz$
S_4	$y=2$	\vec{j}	$dxdz$
S_5	$z=2$	\vec{k}	$-dxdy$

Along OA, $y=0$.

$$= - \iint_{OA} (y\vec{i} + \vec{j} + \vec{k}) \cdot \vec{i} \, dy \, dz$$

$$= - \iint_{\substack{0 \leq x \leq 2 \\ 0 \leq z \leq 2}} dx \, dz$$

$$= - \int_0^2 (x)^2 \frac{1}{2} dz$$

$$= - \int_0^2 (2)^2 dz = - [2(z)]_0^2$$

$$= -2(2) = -4$$

Along AB, $x=2$, $\hat{n} = \vec{i}$, $ds = dydz$

$$= \iint_{AB} (y\vec{i} + \vec{j} + \vec{k}) \cdot \vec{i} \, dy \, dz$$



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$$\begin{aligned}
 &= \iint_{\text{Region}} (yz) dy dz \\
 &= \int_0^2 \left(\frac{y^2}{2} \right)_0^2 dz = \int_0^2 \left(\frac{4}{2} \right) dz \\
 &= 2 \int_0^2 dz = 2 (2)^2 = 8 \\
 &= 2(2) = 4.
 \end{aligned}$$

Along BC: $\vec{r} \cdot \vec{F} = (x - 0) \hat{i} + (y - 0) \hat{j} + (z - 0) \hat{k}$

$$\begin{aligned}
 &= \iint_{BC} (xy \hat{i} + \hat{j} + \hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) dx dy dz \\
 &= \iint_{BC} (x \hat{i}^2 + \hat{j}^2 + \hat{k}^2) dx dy dz \\
 &= \iint_{BC} (x) dx dy dz \\
 &= 2 \int_0^2 dx = 2 (2)^2 = 8
 \end{aligned}$$

Along OC: $x = 0 \rightarrow \hat{n} = -\hat{i}$

$$\begin{aligned}
 &= \iint_{OC} (y \hat{i} + \hat{j} + \hat{k}) \cdot \hat{i} dy dz \\
 &= \iint_{OC} y dy dz \\
 &= - \int_0^2 \left(\frac{y^2}{2} \right)_0^2 dz = - \int_0^2 \left(\frac{4}{2} \right) dz \\
 &= -2 (2)^2 = -8
 \end{aligned}$$

Along z=2: $\hat{n} = \hat{k}$

$$\begin{aligned}
 &= \iint_{z=2} (xy \hat{i} + \hat{j} + \hat{k}) \cdot \hat{k} dx dy \\
 &= \iint_{z=2} (x) dx dy \\
 &= 2 \int_0^2 dx = 2 (2)^2 = 8
 \end{aligned}$$



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Stokes theorem

$$\therefore S_1 + S_2 + S_3 + S_4 + S_5 = -4 + 4 + 4 - 4 + 4$$

~~On RHS.~~ = 4.

LHS:

$$\vec{F} \cdot d\vec{x} = [(y-z+2)\vec{i} + (yz+4)\vec{j} - 4z\vec{k}] \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= (y-z+2)dx + (yz+4)dy - 4zdz$$

$$\int_C \vec{F} \cdot d\vec{x} = \int_{S_1} + \int_{S_2} + \int_{S_3} + \int_{S_4} + \int_{S_5} \vec{F} \cdot d\vec{x}$$

On S_4 , $x=0$, $dx=0$, $z=0$, $dz=0$.

$$\int_{BCO} \vec{F} \cdot d\vec{x} = \int_{S_1} (yz+2)dx + (yz+4)dy - 4zdz$$

$$= \int_{S_1} (4)dy = 4 \int_0^2 dy$$

$$= 4(y)_0^2 = 18.$$

On ∂A , $y=0$, $z=0$, $dy=0$, $dz=0$.

$$\int_{\partial A} 2dx = \int_0^2 2x dx = 2(x)_0^2 = 8.$$

On AB , $x=2$, $z=0$, $dx=0$, $dz=0$.

$$\int_{AB} 4dy = 4 \int_0^2 dy = \left[\frac{4}{2} y \right]_0^2 = 8$$

On BC , $y=2$, $z=0$, $dy=0$, $dz=0$.

$$\int_{BC} 4dx = 4 \int_0^2 dx = 4[x]_0^2 = 8$$

$$\int_{BC} \vec{F} \cdot d\vec{x} = \int_0^2 [4x]_0^2 = -8$$

$$\int_C \vec{F} \cdot d\vec{x} = 18 + 8 - 8 = 18.$$