Continuous Two Dimensional Random Variables

1. Given $f(x,y) = \begin{cases} cx(x-y), & 0 < x < 2, \; -x < y < \\ 0, & \text{otherwise} \end{cases}$

Find i). c ii). $f(x)$ iii). $f(y/x)$

Soln.

i). $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dy\, dx = 1$$

$$\int_0^2\int_{-x}^{x} c\,x(x-y)\, dy\, dx = 1$$

$$c\int_0^2\int_{-x}^{x} (x^2 - xy)\, dy\, dx = 1$$

$$c\int_0^2 \left[x^2 y - x\frac{y^2}{2}\right]_{y=-x}^{x} dx = 1$$

$$c\int_0^2 \left[\left(x^3 - \frac{x^3}{2}\right) - \left(-x^3 - \frac{x^3}{2}\right)\right] dx = 1$$

$$c\int_0^2 \left[x^3 - \frac{x^3}{2} + x^3 + \frac{x^3}{2}\right] dx = 1$$

$$c\int_0^2 2x^3\, dx = 1$$

$$2c\left(\frac{x^4}{4}\right)_0^2 = 1$$

$$\frac{c}{2}(2^4 - 0) = 1$$

$$\frac{16c}{2} = 1$$

$$8c = 1 \Rightarrow c = 1/8$$

$$\therefore f(x,y) = \begin{cases} \frac{1}{8} x(x-y), & 0<x<2, \; -x<y<x \\ 0, & \text{otherwise.} \end{cases}$$

ii). Marginal density function of X (MDF of X)

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_{-x}^{x} \frac{1}{8} x(x-y)\, dy$$

$$= \frac{1}{8} \int_{-x}^{x} (x^2 - xy)\, dy$$

$$= \frac{1}{8} \left[x^2 y - x \frac{y^2}{2} \right]_{y=-x}^{x}$$

$$= \frac{1}{8} \left[\left(x^3 - \frac{x^3}{2}\right) - \left(-x^3 - \frac{x^3}{2}\right) \right]$$

$$= \frac{1}{8} \left[x^3 - \frac{x^3}{2} + x^3 + \frac{x^3}{2} \right]$$

$$= \frac{2x^3}{8}$$

$$f(x) = \frac{x^3}{4}, \quad 0<x<2$$

iii). $f(y/x)$

WKT $\quad f(y/x) = \dfrac{f(x,y)}{f(x)}$

$$= \frac{\frac{1}{8} x(x-y)}{x^3/4}$$

$$= \frac{4/8\,(x^2 - xy)}{x^3}$$

$$= \frac{1}{2}\,\frac{x(x-y)}{x^3}$$

$$F(Y/X) = \frac{x-y}{2x^2}$$

2]. The joint probability density function

$$F(x,y) = \begin{cases} xy^2 + \frac{x^2}{8}, & 0 \le x \le 2 \ \& \ 0 \le y \le 1 \\ 0, & \text{Otherwise} \end{cases}$$

i). $P(X>1/Y<1/2)$ ii). $P(Y<1/2\,/\,X>1)$

iii). $P(X<Y)$ iv). $P(X+Y\le 1)$

Soln.

i). $P(X>1/Y<\frac{1}{2})$

$$= \frac{P(X>1,\ Y<\frac{1}{2})}{P(Y<\frac{1}{2})} \rightarrow (1)$$

Now, $P(X>1,\ Y<\frac{1}{2}) = \int_1^2\int_0^{1/2}\left(xy^2 + \frac{x^2}{8}\right)dy\,dx$

$$= \int_1^2 \left[x\frac{y^3}{3} + \frac{x^2}{8}y\right]_{y=0}^{1/2} dx$$

$$= \int_1^2 \left\{\frac{x}{3}\left(\frac{1}{8}\right) + \frac{x^2}{8}\left(\frac{1}{2}\right) - 0\right\} dx$$

$$= \int_1^2 \left(\frac{x}{24} + \frac{x^2}{16}\right) dx$$

$$= \left(\frac{1}{24}\frac{x^2}{2} + \frac{1}{16}\frac{x^3}{3}\right)_1^2$$

$$= \left(\frac{4}{48} + \frac{8}{48}\right) - \left(\frac{1}{48} + \frac{1}{48}\right)$$

$$= \frac{12}{48} - \frac{2}{48}$$

$$= \frac{10}{48}$$

$$= \frac{5}{24} \rightarrow (2)$$

ii). $P(y < \frac{1}{2} / x > 1)$

$$= \frac{P(x > 1, y < 1/2)}{P(x > 1)}$$

$$P(y < \tfrac{1}{2}) = \int_0^2 \int_0^{1/2} F(x,y)\, dy\, dx$$

$$= \int_0^2 \int_0^{1/2} \left(xy^2 + \frac{x^2}{8}\right) dy\, dx$$

$$= \int_0^2 \left[\frac{xy^3}{3} + \frac{x^2}{8} y\right]_{y=0}^{1/2} dx$$

$$= \int_0^2 \left[\frac{x}{3}\left(\frac{1}{8}\right) + \frac{x^2}{8}\left(\frac{1}{2}\right)\right] dx$$

$$= \int_0^2 \left[\frac{x}{24} + \frac{x^2}{16}\right] dx$$

$$= \left[\frac{x^2}{48} + \frac{x^3}{48}\right]_0^2$$

$$= \frac{4}{48} + \frac{8}{48} = \frac{12}{48}$$

(1) $\Rightarrow P(x>1 / y<\frac{1}{2}) = \frac{5}{24} \cdot \frac{4}{1}$

$= \frac{5}{6}$

ii). $P(y<\frac{1}{2} / x>1)$

$$= \frac{P(x>1, y<\frac{1}{2})}{P(x>1)} \rightarrow (3)$$

Now

$$P(x>1) = \int_1^2\int_0^1 f(x,y)\, dy\, dx$$

$$= \int_1^2\int_0^1 \left(xy^2 + \frac{x^2 y}{8}\right) dy\, dx$$

$$= \int_1^2 \left[\frac{xy^3}{3} + \frac{x^2}{8}\frac{y^2}{2}\right]_{y=0}^{1} dx$$

$$= \int_1^2 \left[\frac{x}{3} + \frac{x^2}{8}\right] dx$$

$$= \left(\frac{x^2}{6} + \frac{x^3}{24}\right)_1^2$$

$$= \left(\frac{4}{6} + \frac{8}{24}\right) - \left(\frac{1}{6} + \frac{1}{24}\right)$$

$$= \frac{16+8-4-1}{24}$$

$$= \frac{19}{24}$$

(3) $\Rightarrow P(y<\frac{1}{2} / x>1) = \frac{5}{24} \times \frac{24}{19} = \frac{5}{19}$

ii i) $P(X < Y)$

$$= \int_0^1 \int_0^y f(x, y)\, dx\, dy$$

$$= \int_0^1 \int_0^y \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$= \int_0^1 \left[\frac{x^2}{2} y^2 + \frac{x^3}{24} \right]_0^y dy$$

$$= \int_0^1 \left[\frac{y^4}{2} + \frac{y^3}{24} \right] dy$$

$$= \left[\frac{y^5}{10} + \frac{y^4}{96} \right]_0^1$$

$$\frac{24 \times 4}{96}$$

$$= \frac{1}{10} + \frac{1}{96}$$

$$= \frac{96+10}{960} = \frac{106}{960}$$

iii). $P(X + Y \leq 1)$

$$= \int_0^1 \int_0^{1-y} f(x, y)\, dx\, dy$$

$$= \int_0^1 \int_0^{1-y} \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$= \int_0^1 \left[\frac{x^2}{2} y^2 + \frac{x^3}{24} \right]_{x=0}^{1-y} dy$$

$$= \int_0^1 \left[\frac{(1-y)^2 y^2}{2} + \frac{(1-y)^3}{24} \right] dy$$

$$= \int_0^1 \left[\frac{(1+y^2-2y)\,y^2}{2} + \frac{(1-y)\,(1-y)^2}{24}\right] dy$$

$$= \int_0^1 \left[\frac{y^2+y^4-2y^3}{2} + \frac{(1-y)\,(1+y^2-2y)}{24}\right] dy$$

$$= \int_0^1 \left[\frac{y^2+y^4-2y^3}{2} + \frac{1+y^2-2y-y-y^3+2y^2}{24}\right] dy$$

$$= \frac{1}{24}\int_0^1 \left[12(y^2+y^4-2y^3)+(1+3y^2-3y-y^3)\right] dy$$

$$= \frac{1}{24}\int_0^1 \left[12y^2+12y^4-24y^3+1+3y^2-3y-y^3\right] dy$$

$$= \frac{1}{24}\int_0^1 \left[12y^4-25y^3+15y^2-3y+1\right] dy$$

$$= \frac{1}{24}\left[\frac{12y^5}{5} - 25\frac{y^4}{4} + 15\frac{y^3}{3} - 3\frac{y^2}{2} + y\right]_{y=0}^{1}$$

$$= \frac{1}{24}\left[\left(\frac{12}{5}-\frac{25}{4}+\frac{15}{3}-\frac{3}{2}+1\right)-0\right]$$

$$\begin{array}{r|l} 2 & 5,\ 4,\ 3,\ 2 \\ \hline & 5,\ 2,\ 3,\ 1 \end{array}$$

$$\frac{20\times 3}{60}$$

$$= \frac{1}{24}\left[\frac{144-375+300-90+60}{60}\right]$$

$$= \frac{39}{1440}$$

$$= 0.027$$

3] The joint PDF of the RV is given by,

$f(x, y) = Kxy\, e^{-(x^2+y^2)}, \quad x>0, y>0.$

find i). K ii). check x & y are independent.

Soln.

i). $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dy\, dx = 1$$

$$\int_0^{\infty}\int_0^{\infty} Kxy\, e^{-(x^2+y^2)}\, dy\, dx = 1$$

$$K\int_0^{\infty}\int_0^{\infty} xy\, e^{-x^2} e^{-y^2}\, dy\, dx = 1$$

Take $x^2 = s$, $ds = 2x\, dx$, $\frac{ds}{2} = x\, dx$ | $y^2 = t$, $2y\,dy = dt$, $y\,dy = \frac{dt}{2}$

Now,

$$K\int_0^{\infty}\int_0^{\infty} e^{-s} e^{-t}\, \frac{dt}{2}\, \frac{ds}{2} = 1$$

$$\frac{K}{4}\int_0^{\infty}\int_0^{\infty} e^{-s} e^{-t}\, dt\, ds = 1$$

$$\frac{K}{4}\int_0^{\infty} e^{-s}\left[\frac{e^{-t}}{-1}\right]_0^{\infty} ds = 1$$

$$-\frac{K}{4}\int_0^{\infty} e^{-s}[0-1]\, ds = 1$$

$$\frac{K}{4}\left(\frac{e^{-s}}{-1}\right)_0^{\infty} = 1$$

$$\frac{-K}{4}(0-1) = 1$$

$$\frac{K}{4} = 1$$

$$\boxed{K = 4}$$

ii). X & Y are independent.

To prove $f(x, y) = f(x) \cdot f(y)$

Now,

$$f(x) = \int_0^\infty f(x, y)\, dy$$

$$= \int_0^\infty 4xy\, e^{-(x^2+y^2)}\, dy$$

$$= 4x e^{-x^2} \int_0^\infty y e^{-y^2}\, dy$$

Take $y^2 = t$

$2y\, dy = dt$

$y\, dy = \frac{dt}{2}$

$$= 4x e^{-x^2} \int_0^\infty e^{-t} \frac{dt}{2}$$

$$= 2x e^{-x^2} \left[\frac{e^{-t}}{-1}\right]_0^\infty$$

$$= -2x e^{-x^2}(0-1)$$

$$f(x) = 2x e^{-x^2}, \quad x > 0$$

$$f(y) = \int_0^\infty f(x, y)\, dx$$

$$= \int_0^\infty 4xy\, e^{-(x^2+y^2)}\, dx$$

$$= 4y e^{-y^2} \int_0^\infty x e^{-x^2}\, dx$$

Take $x^2 = s$

$2x\, dx = ds$

$x\, dx = \frac{ds}{2}$

$$= 4y e^{-y^2} \int_0^\infty e^{-s} \frac{ds}{2}$$

$$= 2y e^{-y^2} \left[\frac{e^{-s}}{-1} \right]_0^\infty$$

$$= -2y e^{-y^2} (0-1)$$

$f(y) = 2y e^{-y^2}, \quad y > 0$

$\therefore f(x) \cdot f(y) = 2x e^{-x^2} \cdot 2y e^{-y^2}$

$= 4xy\, e^{-(x^2+y^2)}$

$= f(x, y)$

$\therefore$ x & y are independent.

4]. If the joint density function of x & y is given by,

$$F(x, y) = \begin{cases} (1-e^{-x})(1-e^{-y}), & x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

To prove x & y are independent.

Soln.

WKT $\quad f(x, y) = \frac{\partial^2}{\partial x\, \partial y} F(x, y)$

$$= \frac{\partial^2}{\partial x \partial y}\left[1 - e^{-x} - e^{-y} + e^{-(x+y)}\right]$$

$$= \frac{\partial}{\partial x}\left[-e^{-y}(-1) + e^{-x} e^{-y}(-1)\right]$$

$$= \frac{\partial}{\partial x}\left[e^{-y} - e^{-x} e^{-y}\right]$$

$$= 0 - e^{-y} e^{-x}(-1)$$

$$= e^{-x} e^{-y}$$

$$f(x,y) = e^{-(x+y)}$$

To prove:

$$f(x,y) = f(x) \cdot f(y)$$

Now,

$$f(x) = \int_0^\infty e^{-(x+y)} \, dy$$

$$= e^{-x} \int_0^\infty e^{-y} \, dy$$

$$= e^{-x} \left(\frac{e^{-y}}{-1}\right)_0^\infty$$

$$= -e^{-x}[0 - 1]$$

$$f(x) = e^{-x}$$

$$f(y) = \int_0^\infty e^{-(x+y)} \, dx$$

$$= \int_0^\infty e^{-x} e^{-y} \, dx$$

$$= e^{-y} \left(\frac{e^{-x}}{-1}\right)_0^\infty$$

$$= -e^{-y}(0 - 1)$$

$$F(y) = e^{-y}$$

$$F(x) \cdot F(y) = e^{-x} \cdot e^{-y}$$
$$= e^{-(x+y)}$$
$$= F(x, y)$$

$\therefore$ x and y are independent.