



Unit - I

Vector Calculus

Scalar quantity ::

A scalar quantity is that which has magnitude and is not related to any direction.

Vector quantity ::

A vector quantity is that which has both magnitude and direction.

Scalar Point function ::

If corresponding to each point P of a region R there corresponds a scalar denoted by $\phi(P)$ or $\phi(x, y, z)$ then ϕ is said to be a scalar point function for the region R .

Example :: The temperature $\phi(P)$ at any point P of a body occupying a certain region is a scalar point function.

Vector Point function ::

If corresponding to each point P of a region R , there corresponds a vector denoted by $F(P)$, then F is said to be a vector point function for the region R .

Example :: The acceleration $F(P)$ of a particle at any time t occupying the position P in a certain region is a vector point function.

Vector Differential operator ::

The vector differential operator ∇ is defined as,

$$\nabla = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} = \sum \vec{i} \frac{\partial}{\partial x}$$



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Gradient of a Scalar Point function:

Let $\phi(x, y, z)$ be a scalar point function and is continuously differentiable then the vector,

$$\begin{aligned}\nabla\phi &= \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \phi \\ &= \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}\end{aligned}$$

is called the gradient of ϕ and is written as $\text{grad } \phi$.

$$\text{ie., } \text{grad } \phi = \nabla\phi$$

Note:

1. $\nabla\phi$ defines a vector field.
2. $\nabla\phi \neq \phi\nabla$ then there will be no 'or' between ϕ and ∇ .

Properties of Gradient:

1. If f and g are two scalar point functions then,

$$\nabla(f \pm g) = \nabla f \pm \nabla g$$

$$\text{(or) } \text{grad}(f \pm g) = \text{grad } f \pm \text{grad } g$$

2. If f and g are two scalar point function then,

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\text{(or) } \text{grad}(fg) = f(\text{grad } g) + g(\text{grad } f)$$

3. If f and g are two scalar point function then,

$$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2} \quad \text{where } g \neq 0$$

$$\text{(or) } \text{grad}\left(\frac{f}{g}\right) = \frac{g(\text{grad } f) - f(\text{grad } g)}{g^2}$$



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4. Gradient of a constant is zero.

$$\text{i.e., } \nabla \phi = 0$$

Problems:

1) Find grad ϕ where $\phi = x^2 + y^2 + z^2$

Solution:

$$\begin{aligned}\nabla \phi &= \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} \\ &= \vec{i} \frac{\partial}{\partial x} (x^2 + y^2 + z^2) + \vec{j} \frac{\partial}{\partial y} (x^2 + y^2 + z^2) \\ &\quad + \vec{k} \frac{\partial}{\partial z} (x^2 + y^2 + z^2) \\ &= \vec{i} (2x) + \vec{j} (2y) + \vec{k} (2z) \\ &= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}\end{aligned}$$

$$\nabla \phi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

2) Find grad ϕ if $\phi = xyz$ at $(1, 1, 1)$

Solution:

$$\begin{aligned}\nabla \phi &= \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} \\ &= \vec{i} \frac{\partial}{\partial x} (xyz) + \vec{j} \frac{\partial}{\partial y} (xyz) + \vec{k} \frac{\partial}{\partial z} (xyz) \\ &= \vec{i} (yz) + \vec{j} (xz) + \vec{k} (xy) \\ \nabla \phi \text{ at } (1, 1, 1) &= \vec{i} (1 \times 1) + \vec{j} (1 \times 1) + \vec{k} (1 \times 1) \\ &= \vec{i} + \vec{j} + \vec{k}\end{aligned}$$

$$\nabla \phi_{(1,1,1)} = \vec{i} + \vec{j} + \vec{k}$$



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(3) Find grad ϕ where $\phi = 3x^2y - y^3z^2$ at $(1, 1, 1)$.

Solution:

$$\begin{aligned}\nabla\phi &= \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z} \\ &= \vec{i} \frac{\partial}{\partial x} (3x^2y - y^3z^2) + \vec{j} \frac{\partial}{\partial y} (3x^2y - y^3z^2) \\ &\quad + \vec{k} \frac{\partial}{\partial z} (3x^2y - y^3z^2)\end{aligned}$$

$$\begin{aligned}&= \vec{i} (6xy) + \vec{j} (3x^2 - 3y^2z^2) + \vec{k} (-2y^3z) \\ \nabla\phi \text{ at } (1, 1, 1) &= \vec{i} (6 \times 1 \times 1) + \vec{j} (3 \times 1^2 - 3 \times 1^2 \times 1^2) + \vec{k} (-2 \times 1^3 \times 1) \\ &= 6\vec{i} + \vec{j} (3 - 3) + \vec{k} (-2) \\ &= 6\vec{i} - 2\vec{k}\end{aligned}$$

$$\boxed{\nabla\phi_{(1,1,1)} = 6\vec{i} - 2\vec{k}}$$

4) If $\phi = \log(x^2 + y^2 + z^2)$ find $\nabla\phi$.

Solution:

$$\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$$

$$\begin{aligned}&= \vec{i} \frac{\partial}{\partial x} [\log(x^2 + y^2 + z^2)] + \vec{j} \frac{\partial}{\partial y} [\log(x^2 + y^2 + z^2)] \\ &\quad + \vec{k} \frac{\partial}{\partial z} [\log(x^2 + y^2 + z^2)]\end{aligned}$$

$$= \vec{i} \frac{1}{x^2 + y^2 + z^2} (2x) + \vec{j} \frac{1}{x^2 + y^2 + z^2} (2y) + \vec{k} \frac{1}{x^2 + y^2 + z^2} (2z)$$

$$= \frac{2}{x^2 + y^2 + z^2} (x\vec{i} + y\vec{j} + z\vec{k})$$

$$\boxed{\nabla\phi = \frac{2\vec{r}}{x^2 + y^2 + z^2}}$$

where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$



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5) Find $\nabla(\log r)$

Solution:

$$\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$$

$$\nabla(\log r) = \vec{i} \frac{\partial}{\partial x} (\log r) + \vec{j} \frac{\partial}{\partial y} (\log r) + \vec{k} \frac{\partial}{\partial z} (\log r)$$

$$= \vec{i} \frac{1}{r} \frac{\partial r}{\partial x} + \vec{j} \frac{1}{r} \frac{\partial r}{\partial y} + \vec{k} \frac{1}{r} \frac{\partial r}{\partial z} \rightarrow \textcircled{1}$$

$$\text{Since } r = \sqrt{x^2 + y^2 + z^2}$$

$$\frac{\partial r}{\partial x} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \cdot \cancel{2x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}$$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$$

Similarly, $\boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$ and $\boxed{\frac{\partial r}{\partial z} = \frac{z}{r}}$

Substitute these values in $\textcircled{1}$,

$$\nabla(\log r) = \vec{i} \left(\frac{1}{r}\right) \left(\frac{x}{r}\right) + \vec{j} \left(\frac{1}{r}\right) \left(\frac{y}{r}\right) + \vec{k} \left(\frac{1}{r}\right) \left(\frac{z}{r}\right)$$

$$= \vec{i} \frac{x}{r^2} + \vec{j} \frac{y}{r^2} + \vec{k} \frac{z}{r^2}$$

$$= \frac{1}{r^2} [x\vec{i} + y\vec{j} + z\vec{k}]$$

$$= \frac{\vec{r}}{r^2}$$

$$\boxed{\nabla(\log r) = \frac{\vec{r}}{r^2}}$$

$$r = \sqrt{x^2 + y^2 + z^2} \quad (1, 2, 3)$$



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≡ 6) If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ such that $|\vec{r}| = r$ Prove that

i) $\nabla r = \frac{\vec{r}}{r} = \hat{r}$

ii) $\nabla \left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3} = -\frac{\hat{r}}{r^2}$

iii) $\nabla r^n = nr^{n-2} \vec{r}$

iv) $\nabla f(r) = f'(r) \nabla r$

v) $\nabla f(r) \times \vec{r} = 0$

vi) If $\nabla \phi$ is Solenoidal find $\nabla^2 \phi$.

Solution:

i) Given: $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

$$2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$2r \frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$2r \frac{\partial r}{\partial z} = 2z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla r = \vec{i} \frac{\partial r}{\partial x} + \vec{j} \frac{\partial r}{\partial y} + \vec{k} \frac{\partial r}{\partial z}$$

$$= \vec{i} \left(\frac{x}{r}\right) + \vec{j} \left(\frac{y}{r}\right) + \vec{k} \left(\frac{z}{r}\right)$$

$$= \frac{1}{r} (x\vec{i} + y\vec{j} + z\vec{k})$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\nabla r = \frac{\vec{r}}{r} = \hat{r}$$



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$$\text{ii) } \nabla \left(\frac{1}{r} \right) = -\frac{\vec{r}}{r^2}$$

$$\nabla \left(\frac{1}{r} \right) = \vec{i} \frac{\partial}{\partial x} \left(\frac{1}{r} \right) + \vec{j} \frac{\partial}{\partial y} \left(\frac{1}{r} \right) + \vec{k} \frac{\partial}{\partial z} \left(\frac{1}{r} \right)$$

$$= \vec{i} \left(-\frac{1}{r^2} \frac{\partial r}{\partial x} \right) + \vec{j} \left(-\frac{1}{r^2} \frac{\partial r}{\partial y} \right) + \vec{k} \left(-\frac{1}{r^2} \frac{\partial r}{\partial z} \right)$$

$$= -\frac{1}{r^2} \left[\vec{i} \frac{x}{r} + \vec{j} \frac{y}{r} + \vec{k} \frac{z}{r} \right]$$

$$\nabla \left(\frac{1}{r} \right) = -\frac{1}{r^3} (\vec{r})$$

$$= -\frac{1}{r^2} \left(\frac{\vec{r}}{r} \right)$$

$$= -\frac{\vec{r}}{r^2}$$

$$\nabla \left(\frac{1}{r} \right) = -\frac{\vec{r}}{r^3} = -\frac{\vec{r}}{r^2}$$

$$\text{iii) } \nabla r^n = n r^{n-2} \vec{r}$$

$$\nabla r^n = \left(\vec{i} \frac{\partial r^n}{\partial x} + \vec{j} \frac{\partial r^n}{\partial y} + \vec{k} \frac{\partial r^n}{\partial z} \right)$$

$$= \vec{i} n r^{n-1} \frac{\partial r}{\partial x} + \vec{j} n r^{n-1} \frac{\partial r}{\partial y} + \vec{k} n r^{n-1} \frac{\partial r}{\partial z}$$

$$= n r^{n-1} \left[\vec{i} \frac{x}{r} + \vec{j} \frac{y}{r} + \vec{k} \frac{z}{r} \right]$$

$$= \frac{n r^{n-1}}{r} [x \vec{i} + y \vec{j} + z \vec{k}]$$

$$= n r^{n-2} \vec{r}$$

$$\nabla r^n = n r^{n-2} \vec{r}$$



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$$iv) \nabla f(r) \times \vec{r} = 0$$

$$iv) \nabla f(r) = f'(r) \nabla r$$

$$\nabla f(r) = \vec{i} \frac{\partial}{\partial x} f(r) + \vec{j} \frac{\partial}{\partial y} f(r) + \vec{k} \frac{\partial}{\partial z} f(r)$$
$$= \vec{i} f'(r) \frac{\partial r}{\partial x} + \vec{j} f'(r) \frac{\partial r}{\partial y} + \vec{k} f'(r) \frac{\partial r}{\partial z}$$

$$= f'(r) \left[\vec{i} \frac{x}{r} + \vec{j} \frac{y}{r} + \vec{k} \frac{z}{r} \right]$$

$$= \frac{f'(r)}{r} (x\vec{i} + y\vec{j} + z\vec{k})$$

$$= \frac{f'(r)}{r} \vec{r}$$

$$\nabla f(r) = \frac{f'(r)}{r} \vec{r}$$

$$v) \nabla f(r) \times \vec{r} = 0$$

$$\nabla f(r) \times \vec{r} = \frac{f'(r)}{r} \vec{r} \times \vec{r}$$

$$= \frac{1}{r} f'(r) [\vec{r} \times \vec{r}]$$

$$= 0$$

$$(\because \vec{r} \times \vec{r} = 0)$$

$$\nabla f(r) \times \vec{r} = 0$$

$$vi) \nabla^2 \phi = 0$$

$$\nabla^2 \phi = \nabla(\nabla \phi)$$

$$= \nabla(0)$$

$$= 0$$

$$\nabla^2 \phi = 0$$

($\because \nabla \phi$ is Solenoidal $\nabla \phi = 0$)



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Level Surface: Important Results:

Unit Normal:

A Unit normal to the given surface ϕ at the point

$$\text{is } \frac{\nabla\phi}{|\nabla\phi|}$$

Directional Derivative:

The directional derivative of ϕ in the direction \vec{a} is given

by,

$$\nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} \quad (\text{or}) \quad \nabla\phi \cdot \hat{n} \quad \text{where} \quad \hat{n} = \frac{\vec{a}}{|\vec{a}|}$$

The directional derivative is maximum in the direction of the normal to the given surface. Its maximum value is $|\nabla\phi|$.

Angle between two surfaces:

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1| \cdot |\nabla\phi_2|}$$

Note:

If the surfaces cut orthogonally then

$$\nabla\phi_1 \cdot \nabla\phi_2 = 0$$

Problems:

1) Find a unit normal to the surface $x^2y + 2xz = 4$ at $(2, -2, 3)$

Solution:

$$\phi: x^2y + 2xz - 4$$

$$\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$$



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$$\begin{aligned} \vec{n} &= \vec{i} \frac{\partial}{\partial x} (xy + 2xz - 4) + \vec{j} \frac{\partial}{\partial y} (x^2y + 2xz - 4) \\ &+ \vec{k} \frac{\partial}{\partial z} (x^2y + 2xz - 4) \end{aligned}$$

$$= \vec{i}(2xy + 2z) + \vec{j}(x^2) + \vec{k}(2x)$$

$$\nabla\phi(2,-2,3) = \vec{i}(-8+6) + \vec{j}(4) + \vec{k}(4)$$

$$= -2\vec{i} + 4\vec{j} + 4\vec{k}$$

$$|\nabla\phi| = \sqrt{4+16+16} = \sqrt{36} = 6$$

Unit normal to the given Surface at $(2, -2, 3)$

$$= \frac{\nabla\phi}{|\nabla\phi|} = \frac{-2\vec{i} + 4\vec{j} + 4\vec{k}}{6}$$

$$= \frac{1}{3}(-\vec{i} + 2\vec{j} + 2\vec{k})$$

$$\text{Unit normal} = \frac{1}{3}(-\vec{i} + 2\vec{j} + 2\vec{k})$$

2) Find the unit vector normal to $x^2 - y^2 + z = 2$ at $(1, -1, 2)$

$$\text{Sol} \therefore \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i} + 2\vec{j} + \vec{k}}{3}$$

3) Find the unit vector normal to $x^2 + xy + z^2 = 4$ at $(1, -1, 2)$

$$\text{Sol} \therefore \frac{\nabla\phi}{|\nabla\phi|} = \frac{\vec{i} + \vec{j} + 4\vec{k}}{\sqrt{18}}$$



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4) Find the directional derivative of the function $x^2 + 2xy$ at $(1, -1, 3)$ in the direction $\vec{i} + 2\vec{j} + 2\vec{k}$.

Sol: $\phi = x^2 + 2xy$

$$\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$$

$$= \vec{i} \frac{\partial}{\partial x}(x^2 + 2xy) + \vec{j} \frac{\partial}{\partial y}(x^2 + 2xy) + \vec{k} \frac{\partial}{\partial z}(x^2 + 2xy)$$

$$= \vec{i}(2x + 2y) + \vec{j}(2x) + \vec{k}(0)$$

$$= \vec{i}(2x + 2y) + \vec{j}(2x)$$

$$\nabla\phi_{(1, -1, 3)} = \vec{i}(2(1) + 2(-1)) + \vec{j}(2(1))$$
$$= \vec{i}(2 - 2) + \vec{j}(2)$$

$$\nabla\phi_{(1, -1, 3)} = 2\vec{j}$$

Given: $\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$

$$|\vec{a}| = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$\hat{n} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{i} + 2\vec{j} + 2\vec{k}}{3}$$

$$\nabla\phi \cdot \hat{n} = 2\vec{j} \cdot \left[\frac{\vec{i} + 2\vec{j} + 2\vec{k}}{3} \right]$$

$$= \frac{4}{3}$$

$$\nabla\phi \cdot \hat{n} = \frac{4}{3}$$



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5) Find the directional derivative of $xy + yz + zx$ at $(1,1,1)$ in the direction $\vec{i} + \vec{j}$.

Sol: $\therefore 2\sqrt{2}$

6) Find the directional derivative of $3x^2 + 2y - 3z$ at $(1,1,1)$ in the direction $2\vec{i} + 2\vec{j} - \vec{k}$.

Sol: $\frac{19}{3}$

7) What is the greatest rate of increase of $\phi = xyz^2$ at $(1,0,3)$?

Sol: Let $\phi = xyz^2$

$$\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$$

$$= \vec{i} \frac{\partial}{\partial x} (xyz^2) + \vec{j} \frac{\partial}{\partial y} (xyz^2) + \vec{k} \frac{\partial}{\partial z} (xyz^2)$$

$$= \vec{i} (yz^2) + \vec{j} (xz^2) + \vec{k} (2xyz)$$

$$\nabla\phi_{(1,0,3)} = \vec{i} (0) + \vec{j} (9) + \vec{k} (0)$$
$$= 9\vec{j}$$

$$\boxed{\nabla\phi_{(1,0,3)} = 9\vec{j}}$$

Maximum (or) Greatest rate of increase = $|\nabla\phi|$

$$= \sqrt{9^2}$$

$$= \sqrt{81}$$

$$= 9$$

$$\boxed{\text{Greatest rate of increase} = 9}$$



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8) In what direction from the point $(1, -1, 2)$ is the directional derivative of $\phi = x^2 y z^2$ a maximum? What is the magnitude of this maximum?

Sol.: $\phi = x^2 y z^2$

$$\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$$

$$= \vec{i} \frac{\partial}{\partial x} (x^2 y z^2) + \vec{j} \frac{\partial}{\partial y} (x^2 y z^2) + \vec{k} \frac{\partial}{\partial z} (x^2 y z^2)$$

$$= 2xyz^2 \vec{i} + 2x^2 y z^2 \vec{j} + 2x^2 y z^2 \vec{k}$$

$\nabla\phi_{(1,-1,2)} = 16\vec{i} - 16\vec{j} + 12\vec{k}$ is the directional derivative.

Magnitude is $|\nabla\phi| = \sqrt{16^2 + 16^2 + 12^2}$
 $= \sqrt{256 + 256 + 144}$
 $= \sqrt{656}$

Magnitude = $|\nabla\phi| = \sqrt{656}$

9) Find the directional derivative of $\phi = x^2 y z^2$ at the point $(1, 1, 1)$ along the normal to the surface $x^2 + xy + z^2 = 3$ at the point $(1, 1, 1)$.

Sol.: $\nabla\phi$ is normal to the surface $x^2 + xy + z^2 = 3$

$$\nabla\phi = \vec{i} \frac{\partial\phi}{\partial x} + \vec{j} \frac{\partial\phi}{\partial y} + \vec{k} \frac{\partial\phi}{\partial z}$$

$$= \vec{i} \frac{\partial}{\partial x} (x^2 + xy + z^2 - 3) + \vec{j} \frac{\partial}{\partial y} (x^2 + xy + z^2 - 3) + \vec{k} \frac{\partial}{\partial z} (x^2 + xy + z^2 - 3)$$

$$= \vec{i} (2x + y) + \vec{j} (x) + \vec{k} (2z)$$

$\nabla\phi_{(1,1,1)} = 3\vec{i} + \vec{j} + 2\vec{k}$



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To find the directional derivative of $\phi = xyz^3$ at $(1,1,1)$ in the direction $\vec{a} = 3\vec{i} + \vec{j} + 2\vec{k}$.

$$\nabla\phi = \vec{i} \frac{\partial}{\partial x} (xyz^3) + \vec{j} \frac{\partial}{\partial y} (xyz^3) + \vec{k} \frac{\partial}{\partial z} (xyz^3)$$

$$= \vec{i} (yz^3) + \vec{j} (2xyz^2) + \vec{k} (3xy^2z^2)$$

$$\nabla\phi(1,1,1) = \vec{i}(1) + \vec{j}(2) + \vec{k}(3)$$

$$\nabla\phi(1,1,1) = \vec{i} + 2\vec{j} + 3\vec{k}$$

$$\text{Directional derivative} = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= (\vec{i} + 2\vec{j} + 3\vec{k}) \cdot \frac{(3\vec{i} + \vec{j} + 2\vec{k})}{\sqrt{9+1+4}}$$

$$= \frac{3+2+6}{\sqrt{14}}$$

$$= \frac{11}{\sqrt{14}}$$

$$\text{Directional derivative} = \frac{11}{\sqrt{14}}$$

10) Find the angle between the surfaces $x^2 + y^2 + z^2 = 5$ and $x^2 + y^2 + z^2 - 2x = 5$ at $(0,1,2)$.

Sol: let $\phi_1 = x^2 + y^2 + z^2 - 5$

$$\frac{\partial\phi_1}{\partial x} = 2x$$

$$\frac{\partial\phi_1}{\partial y} = 2y$$

$$\frac{\partial\phi_1}{\partial z} = 2z$$

$$\phi_2 = x^2 + y^2 + z^2 - 2x - 5$$

$$\frac{\partial\phi_2}{\partial x} = 2x - 2$$

$$\frac{\partial\phi_2}{\partial y} = 2y$$

$$\frac{\partial\phi_2}{\partial z} = 2z$$



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$$\nabla\phi_1 = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\nabla\phi_2 = (2x-2)\vec{i} + 2y\vec{j} + 2z\vec{k}$$
$$= 2x\vec{i} - 2\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\nabla\phi_1(0,1,2) = 2(0)\vec{i} + 2(1)\vec{j} + 2(2)\vec{k}$$
$$= 0 + 2\vec{j} + 4\vec{k}$$

$$\nabla\phi_1(0,1,2) = 2\vec{j} + 4\vec{k}$$

$$|\nabla\phi_1| = \sqrt{2^2 + 4^2} = \sqrt{4 + 16} = \sqrt{20}$$

$$|\nabla\phi_1| = \sqrt{20}$$

$$\nabla\phi_2(0,1,2) = -2\vec{i} + 2\vec{j} + 4\vec{k}$$

$$\nabla\phi_2(0,1,2) = -2\vec{i} + 2\vec{j} + 4\vec{k}$$

$$|\nabla\phi_2| = \sqrt{2^2 + 2^2 + 4^2}$$

$$|\nabla\phi_2| = \sqrt{2^2 + 2^2 + 4^2} = \sqrt{4 + 4 + 16} = \sqrt{24} = \sqrt{24}$$

$$|\nabla\phi_2| = \sqrt{24}$$

Angle between the surfaces,

$$\cos \theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1| |\nabla\phi_2|}$$

$$= \frac{(2\vec{j} + 4\vec{k}) \cdot (-2\vec{i} + 2\vec{j} + 4\vec{k})}{\sqrt{20} \cdot \sqrt{24}}$$

$$= \frac{4 + 16}{\sqrt{20} \cdot \sqrt{24}}$$

$$= \frac{4 + 16}{\sqrt{20} \cdot \sqrt{24}}$$



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$$\begin{aligned}\cos \theta &= \frac{4+16}{\sqrt{20} \cdot \sqrt{24}} \\ &= \frac{20}{\sqrt{20} \cdot \sqrt{24}} \\ &= \frac{\sqrt{20} \cdot \sqrt{20}}{\sqrt{20} \cdot \sqrt{24}} \\ &= \sqrt{\frac{20}{24}}\end{aligned}$$

$$\cos \theta = \sqrt{\frac{5}{6}}$$

$$\theta = \cos^{-1} \sqrt{\frac{5}{6}}$$

1) Find the angle between the surfaces $x \log z = y^2 - 1$ and $x^2 y = 2 - z$ at the point $(1, 1, 1)$.

Sol: $\phi_1 = x \log z - y^2 + 1$

$$\frac{\partial \phi_1}{\partial x} = \log z$$

$$\frac{\partial \phi_1}{\partial y} = -2y$$

$$\frac{\partial \phi_1}{\partial z} = \frac{x}{z}$$

$$\nabla \phi_1 = \log z \vec{i} - 2y \vec{j} + \frac{x}{z} \vec{k}$$

$$\nabla \phi_1(1,1,1) = -2\vec{j} + \vec{k}$$

$$\phi_2 = x^2 y - 2 + z$$

$$\frac{\partial \phi_2}{\partial x} = 2xy$$

$$\frac{\partial \phi_2}{\partial y} = x^2$$

$$\frac{\partial \phi_2}{\partial z} = 1$$

$$\nabla \phi_2 = (2xy)\vec{i} + x^2\vec{j} + \vec{k}$$

$$\nabla \phi_2 = 2\vec{i} + \vec{j} + \vec{k}$$

$$\nabla \phi_2(1,1,1) = 2\vec{i} + \vec{j} + \vec{k}$$



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$$|\nabla\phi_1| = \sqrt{4+1} = \sqrt{5}$$

$$|\nabla\phi_2| = \sqrt{4+1+1} = \sqrt{6}$$

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1| \cdot |\nabla\phi_2|}$$

$$= \frac{(-2\vec{j} + \vec{k}) \cdot (2\vec{i} + \vec{j} + \vec{k})}{\sqrt{5} \cdot \sqrt{6}}$$

$$= \frac{-2+1}{\sqrt{30}}$$

$$\cos\theta = -\frac{1}{\sqrt{30}}$$

$$\theta = \cos^{-1}\left(-\frac{1}{\sqrt{30}}\right)$$

12) Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 2$ at $(2, -1, 2)$.

Sol: $\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$

13) Find a and b such that the surfaces $ax^2 + byz = (a+2)x$ and $4x^2y + z^3 = 4$ cuts orthogonally at $(1, -1, 2)$.

Sol: Let $\phi_1 = ax^2 + byz - (a+2)x$

$$\phi_2 = 4x^2y + z^3 - 4$$

$$\nabla\phi_1 = \vec{i} \frac{\partial}{\partial x} (ax^2 + byz - (a+2)x) + \vec{j} \frac{\partial}{\partial y} (ax^2 + byz - (a+2)x) + \vec{k} \frac{\partial}{\partial z} (ax^2 + byz - (a+2)x)$$

$$= \vec{i}(2ax - a - 2) + \vec{j}(bz) + \vec{k}(by)$$



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$$\nabla\phi_1 = \vec{i}(2ax - a - 2) + \vec{j}(bz) + \vec{k}(by)$$

$$\begin{aligned}\nabla\phi_1(1, -1, 2) &= \vec{i}(2a(1) - a - 2) + \vec{j}(b(2)) + \vec{k}(b(-1)) \\ &= \vec{i}(2a - a - 2) + \vec{j}(2b) + \vec{k}(-b) \\ &= \vec{i}(a - 2) + \vec{j}(2b) + \vec{k}(-b)\end{aligned}$$

$$\nabla\phi_1(1, -1, 2) = (a-2)\vec{i} + 2b\vec{j} - b\vec{k}$$

$$\begin{aligned}\nabla\phi_2 &= \vec{i}\frac{\partial}{\partial x}(4x^2y + z^3 - 4) + \vec{j}\frac{\partial}{\partial y}(4x^2y + z^3 - 4) \\ &\quad + \vec{k}\frac{\partial}{\partial z}(4x^2y + z^3 - 4)\end{aligned}$$

$$= \vec{i}(8xy) + \vec{j}(4x^2) + \vec{k}(3z^2)$$

$$\begin{aligned}\nabla\phi_2(1, -1, 2) &= \vec{i}(8(1)(-1)) + \vec{j}(4(1)) + \vec{k}(3(2)^2) \\ &= -8\vec{i} + 4\vec{j} + 12\vec{k}\end{aligned}$$

$$\nabla\phi_2(1, -1, 2) = -8\vec{i} + 4\vec{j} + 12\vec{k}$$

Since the surfaces cut orthogonally,

$$\nabla\phi_1 \cdot \nabla\phi_2 = 0$$

$$[(a-2)\vec{i} + 2b\vec{j} - b\vec{k}] \cdot [-8\vec{i} + 4\vec{j} + 12\vec{k}] = 0$$

$$-8a + 16 - 8b + 12b = 0$$

$$-8a + 4b = -16$$

$$2a - b = 4 \rightarrow \textcircled{1}$$

Since the point $(1, -1, 2)$ lies on ϕ_1 ,

$$a - 2b - (a + 2) = 0 \Rightarrow \boxed{b = -1}$$

$$\text{Sub } b = -1 \text{ in equ } \textcircled{1} \Rightarrow \boxed{a = \frac{3}{2}}$$