14) Find the values of a and b so that the surface $ax^3 - by^2z = (a+3)x^2$ and $4x^2y - z^3 = 11$ may cut orthogonally at $(2,-1,-3)$.

Sol: $a = \frac{-7}{3}$ and $b = \frac{64}{9}$

Divergence of a Vector Point Function:

Let $\vec{F}$ be any given continuously differentiable vector point function then the divergence of $\vec{F}$ is defined as,

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot \vec{F}$$

$$= \vec{i}\,\frac{\partial \vec{F}}{\partial x} + \vec{j}\,\frac{\partial \vec{F}}{\partial y} + \vec{k}\,\frac{\partial \vec{F}}{\partial z}$$

Note:

1. $\nabla \cdot \vec{F}$ is a scalar point function.
2. If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ be a continuously differentiable vector point function then

$$\boxed{\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

Solenoidal Vector:

A vector $\vec{F}$ is said to be solenoidal vector if $\operatorname{div} \vec{F} = 0$.

Curl of a Vector Point function :

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ be any given continuously differentiable vector point function, the curl or rotation of $\vec{F}$ is defined as,

$$\text{curl } \vec{F} = \nabla \times \vec{F}$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \vec{F}$$

$$= \vec{i} \times \frac{\partial \vec{F}}{\partial x} + \vec{j} \times \frac{\partial \vec{F}}{\partial y} + \vec{k} \times \frac{\partial \vec{F}}{\partial z}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Note : $\nabla \times \vec{F}$ is a vector point function.

Irrotational Vector :

A vector $\vec{F}$ is said to be irrotational if

$$\nabla \times \vec{F} = 0$$

ie.) $$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = 0$$

Conservative Vector Field :

If a vector point function $\vec{F}$ is expressible as the gradient of a scalar point function ϕ, then $\vec{F}$ is conservative. ie.,) $\vec{F}$ is conservative if $\vec{F} = \nabla\phi$. Here ϕ is called scalar potential.

$\vec{F}$ is conservative force if curl $\vec{F} = 0$

Problems:

1) Prove that $\text{Curl}(\nabla\phi) = 0$ (or) $\nabla \times \nabla\phi = 0$

Sol:

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\text{Curl}(\nabla\phi) = \nabla\times\nabla\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial y\partial z}\right) - \vec{j}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial x\partial z}\right) + \vec{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial x\partial y}\right)$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(0)$$

$$\boxed{\text{Curl}(\nabla\phi) = 0}$$

2) Prove that $\text{div}(\text{curl}\,\vec{F}) = 0$ (or) $\nabla\cdot(\nabla\times\vec{F}) = 0$.

Sol:

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \quad \text{if } \vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

$$\nabla\times\vec{F} = \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\nabla\cdot\nabla\times\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left[\vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]$$

$$\nabla\cdot\nabla\times\vec{F} = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= 0$$

$$\boxed{\nabla\cdot\nabla\times\vec{F} = 0}$$

---

3) Show that Curl grad $f = 0$ (or) $\nabla\times\nabla f = 0$

Sol:

$$\text{Curl grad } f = \nabla\times\nabla f$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial y\partial z}\right] - \vec{j}\left[\frac{\partial^2 f}{\partial x\partial z} - \frac{\partial^2 f}{\partial x\partial z}\right] + \vec{k}\left[\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial x\partial y}\right]$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(0)$$

$$= 0$$

$$\boxed{\text{Curl grad } f = 0}$$

---

4) If $\nabla V = y\vec{i} + z\vec{j} + x\vec{k}$ what is the directional derivative of V at the point $(1,2,3)$ in the direction $3\vec{i}+4\vec{j}+5\vec{k}$.

Sol: $\nabla V = y\vec{i} + z\vec{j} + x\vec{k}$

$$\nabla v_{(1,2,3)} = 2\vec{i}+3\vec{j}+\vec{k}$$

$$\vec{a} = 3\vec{i}+4\vec{j}+5\vec{k}$$

$$|\vec{a}| = \sqrt{9+16+25} = \sqrt{50}$$

$$\text{Directional derivative} = \nabla v \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= (2\vec{i}+3\vec{j}+\vec{k}) \cdot \frac{(3\vec{i}+4\vec{j}+5\vec{k})}{\sqrt{50}}$$

$$= \frac{6+12+5}{\sqrt{50}}$$

$$= \frac{23}{\sqrt{50}}$$

$$\boxed{\text{Directional derivative} = \frac{23}{\sqrt{50}}}$$

5) Prove that $\operatorname{div}(\vec{u}\times\vec{v}) = \vec{v}\cdot \operatorname{curl}\vec{u} - \vec{u}\operatorname{curl}\vec{v}$

Sol:
$$\operatorname{div}(\vec{u}\times\vec{v}) = \sum \vec{i}\frac{\partial}{\partial x}(\vec{u}\times\vec{v})$$

$$= \sum \vec{i}\left[\vec{u}\times\frac{\partial \vec{v}}{\partial x} + \frac{\partial \vec{u}}{\partial x}\times\vec{v}\right]$$

$$= \sum \vec{i}\cdot\left(\vec{u}\times\frac{\partial \vec{v}}{\partial x}\right) + \sum \vec{i}\left(\frac{\partial \vec{u}}{\partial x}\times\vec{v}\right)$$

$$= \left(\sum \vec{i}\times\frac{\partial \vec{u}}{\partial x}\right)\cdot\vec{v} - \left(\sum \vec{i}\times\frac{\partial \vec{v}}{\partial x}\right)\cdot\vec{u}$$

$$= \operatorname{curl}\vec{u}\cdot\vec{v} - \operatorname{curl}\vec{v}\cdot\vec{u}$$

$$= \vec{v}\operatorname{curl}\vec{u} - \vec{u}\operatorname{curl}\vec{v}$$

$$\boxed{\operatorname{div}(\vec{u}\times\vec{v}) = \vec{v}\cdot\operatorname{curl}\vec{u} - \vec{u}\cdot\operatorname{curl}\vec{v}}$$

7) Find the constants a, b, c so that the vector $\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ is irrotational.

Sol: $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$

$= \vec{i}\left[\frac{\partial}{\partial y}[4x+cy+2z] - \frac{\partial}{\partial z}(bx-3y-z)\right]$

$-\vec{j}\left[\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right]$

$+\vec{k}\left[\frac{\partial}{\partial x}[bx-3y-z] - \frac{\partial}{\partial y}(x+2y+az)\right]$

$= \vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2)$

Given : $\vec{F}$ is irrotational

ie.,) $\nabla \times \vec{F} = 0$

$\vec{i}(c+1) - \vec{j}(4-a) + \vec{k}(b-2) = 0$

$c+1 = 0 \Rightarrow \boxed{c=-1}$

$4-a = 0 \Rightarrow \boxed{a=4}$

$b-2 = 0 \Rightarrow \boxed{b=2}$

---

8) Find "a" so that the vector $\vec{A} = (ax^2-y^2+x)\vec{i} - (2xy+y)\vec{j}$ is irrotational.

Sol: Given : $\vec{A}$ is irrotational.

$\nabla \times \vec{A} = 0$

$$\nabla\times\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax^2-y^2+x & -(2xy+y) & 0 \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(-2y+2y)$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(0)$$

$$= 0$$

$$\boxed{\nabla\times\vec{A} = 0}$$

$\therefore$ 'a' is arbitrary.

---

9) Prove $\vec{F} = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$ is irrotational and find its scalar potential ϕ such that $\vec{F} = \nabla\phi$.

Sol:

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(3xz^2) - \frac{\partial}{\partial z}(2y\sin x - 4)\right] - \vec{j}\left[\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial z}(y^2\cos x + z^3)\right] + \vec{k}\left[\frac{\partial}{\partial x}(2y\sin x - 4) - \frac{\partial}{\partial y}(y^2\cos x + z^3)\right]$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$$= 0$$

$\therefore \vec{F}$ is irrotational.

7 To find ϕ:

$$\nabla\phi = (y^2\cos x + z^2)\vec{i} + (2y\sin x - 4)\vec{j} + 3xz^2\vec{k}$$

We know that $\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$.

$$\frac{\partial\phi}{\partial x} = y^2\cos x + z^3 \Rightarrow \phi = y^2\sin x + z^3x + f(y,z)$$

$$\frac{\partial\phi}{\partial y} = 2y\sin x - 4 \Rightarrow \phi = y^2\sin x - 4y + f(x,z)$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 \Rightarrow \phi = xz^3 + f(x,y)$$

---

10) Show that $\vec{F} = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}$ is irrotational. Find ϕ such that $\vec{F} = \nabla\phi$.

Sol: $\phi = 3x^2y + xz^3 - yz + c$

11) If $\nabla\phi = yz\vec{i} + xz\vec{j} + xy\vec{k}$ then find ϕ.

Sol: $\phi = xyz + c$

12) Prove that $\operatorname{div}\hat{r} = \frac{2}{r}$.

Sol: $\operatorname{div}\hat{r} = \nabla\cdot\left(\frac{\vec{r}}{r}\right)$

$$= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot\left(\frac{x\vec{i} + y\vec{j} + z\vec{k}}{r}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$= \frac{1}{r} - \frac{1}{r^2}\cdot x\frac{\partial r}{\partial x} + \frac{1}{r} - \frac{1}{r^2}y\frac{\partial r}{\partial y} + \frac{1}{r} - \frac{1}{r^2}z\frac{\partial r}{\partial z}$$

$$= \frac{3}{r} - \frac{1}{r^2}\left[x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right]$$

Now, $r^2 = x^2+y^2+z^2$

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$2r\frac{\partial r}{\partial z} = 2z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r}$$

Now, $\operatorname{div} \hat{r} = \frac{3}{r} - \frac{1}{r^2}\left[x\cdot\frac{x}{r} + y\cdot\frac{y}{r} + z\cdot\frac{z}{r}\right]$

$$= \frac{3}{r} - \frac{1}{r^2}\left[\frac{x^2+y^2+z^2}{r}\right]$$

$$= \frac{3}{r} - \frac{1}{r^2}\cdot\frac{r^2}{r}$$

$$= \frac{3}{r} - \frac{1}{r}$$

$$= \frac{2}{r}$$

$$\boxed{\operatorname{div}\hat{r} = \frac{2}{r}}$$

13) Prove that $(\text{curl curl } \vec{F}) = \nabla(\operatorname{div}\vec{F}) - \nabla^2\vec{F}$.

Sol: $\nabla\times(\nabla\times\vec{F}) = (\nabla\cdot\vec{F})\cdot\nabla - (\nabla\cdot\nabla)\vec{F}$

$$\left[\because \vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\cdot\vec{c}\right]$$

$$= \nabla\cdot(\nabla\cdot\vec{F}) - \nabla^2\vec{F}$$

$$\boxed{\nabla\times(\nabla\times\vec{F}) = \nabla(\operatorname{div}\vec{F}) - \nabla^2\vec{F}.}$$