14) Prove that $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$ is a conservative force and hence find ϕ so that $\vec{F} = \nabla\phi$.

Sol: $\vec{F}$ is a conservative force ie.,) curl $\vec{F} = 0$

and $\phi = x^2y + xz^3 + c$ where c is any arbitrary constant.

Green's theorem in a plane:

If R is a closed region of the xy-plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R then

$$\boxed{\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}$$

where C is a curve traversed in the anticlockwise direction.

Problems:

1) Evaluate by Green's theorem $\int_C (xy + x^2)\,dx + (x^2 + y^2)\,dy$ where C is the square formed by $x = -1, x = 1, y = -1, y = 1$.

Sol: Let R be the region enclosed by C.

By Green's theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Here $M = xy + x^2 \Rightarrow \frac{\partial M}{\partial y} = x$

$N = x^2 + y^2 \Rightarrow \frac{\partial N}{\partial x} = 2x$

$$\int_C (xy+x^2)\,dx + (x^2+y^2)\,dy = \iint_R (2x-x)\,dx\,dy$$

$$= \int_{-1}^{1}\int_{-1}^{1} x\,dx\,dy$$

$$= \int_{-1}^{1}\left[\frac{x^2}{2}\right]_{-1}^{1} dy$$

$$= 0$$

$$\boxed{\int_C (xy+x^2)\,dx + (x^2+y^2)\,dy = 0}$$

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2) Evaluate by Green's theorem $\int_C e^{-x}(\sin y\,dx + \cos y\,dy)$ where c is the rectangle with vertices $(0,0)$, $(\pi,0)$, $(\pi,\frac{\pi}{2})$, $(0,\frac{\pi}{2})$.

Sol:. Let R be the region enclosed by C.

By Green's theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Here $M = e^{-x}\sin y \Rightarrow \frac{\partial M}{\partial y} = e^{-x}\cos y$

$N = e^{-x}\cos y \Rightarrow \frac{\partial N}{\partial x} = -e^{-x}\cos y$

$$\int_C e^{-x}(\sin y\,dx + \cos y\,dy) = \iint_R (-e^{-x}\cos y - e^{-x}\cos y)\,dx\,dy$$

$$= \int_0^{\pi/2}\int_0^{\pi} (-2e^{-x}\cos y)\,dx\,dy$$

$$= -2\int_0^{\pi/2}\int_0^{\pi} e^{-x}\cos y\,dx\,dy$$

$$= 2(e^{-\pi}-1)$$

$$\boxed{\int_C e^{-x}(\sin y\,dx + \cos y\,dy) = 2(e^{-\pi}-1)}$$

3) Evaluate by Green's theorem $\int_C (x^2-\cosh y)dx + (y+\sin x)dx$, where C is the rectangle with vertices $(0,0), (\pi,0), (\pi,1), (0,1)$.

Sol: $\pi(\cosh -1)$.

4) Verify Green's theorem in the plane for $\int_C (3x^2-8y^2)dx + (4y-6xy)dy$ where C is the boundary of the region defined by $x=y^2, y=x^2$.

Sol: By Green's theorem,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$$

Given: $\int_C (3x^2-8y^2)dx + (4y-6xy)dy$

Here $M = 3x^2-8y^2 \Rightarrow \frac{\partial M}{\partial y} = -16y$

$N = 4y-6xy \Rightarrow \frac{\partial N}{\partial x} = -6y$

Step : 1

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (-6y + 16y)\,dx\,dy$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} 10y\,dx\,dy$$

$$= \int_0^1 10y\,[x]_{y^2}^{\sqrt{y}}\,dy$$

$$= \int_0^1 10y\,(\sqrt{y} - y^2)\,dy$$

$$= 10\int_0^1 (y^{3/2} - y^3)\,dy = 10\left[\frac{y^{5/2}}{5/2} - \frac{y^4}{4}\right]_0^1$$

$$= 10\left(\frac{2}{5} - \frac{1}{4}\right) = 10\left(\frac{8-5}{20}\right)$$

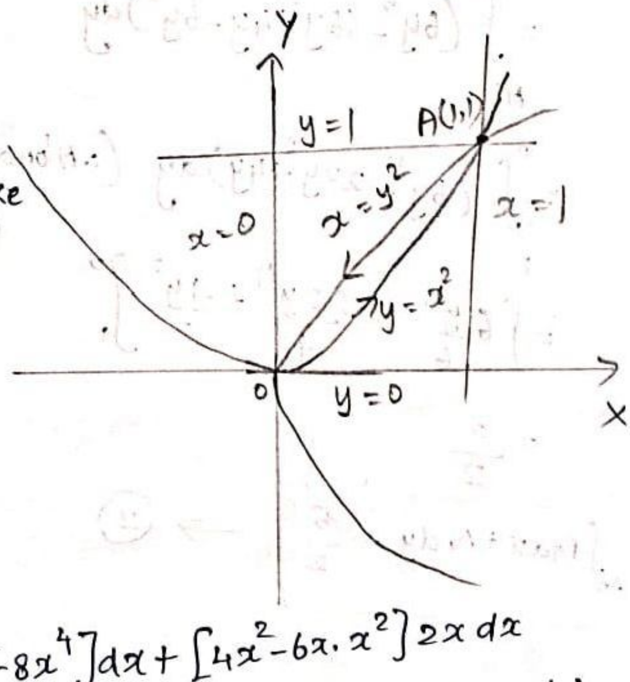
$$\boxed{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \frac{3}{2}} \longrightarrow (1)$$

Step : 2

To evaluate $\int_C M\,dx + N\,dy$ we take C in different Paths

(i) along OA $(y = x^2)$

(ii) along AO $(x = y^2)$



(i) Along OA :-

$$\int_{OA} M\,dx + N\,dy = \int_{OA} [3x^2 - 8x^4]\,dx + [4x^2 - 6x \cdot x^2]\,2x\,dx$$

$$(\because x^2 = y,\ 2x\,dx = dy)$$

$$= \int_0^1 (3x^2 - 8x^4 + 8x^3 - 12x^4)\,dx \quad (\because \text{Along } OA, \; x \text{ varies from } 0 \text{ to } 1)$$

$$= \int_0^1 (-20x^4 + 8x^3 + 3x^2)\,dx$$

$$= \left[-20\frac{x^5}{5} + \frac{8x^4}{4} + \frac{3x^3}{3} \right]_0^1$$

$$= -1$$

$$\boxed{\int_{OA} M\,dx + N\,dy = -1}$$

ii) Along AO :-

$$\int_{AO} M\,dx + N\,dy = \int_{AO} (3y^4 - 8y^2)\,2y\,dy + (4y - 6yy)\,dy$$

$$(\because y^2 = x, \; 2y\,dy = dx)$$

$$= \int_{AO} (6y^5 - 16y^3 + 4y - 6y^3)\,dy$$

$$= \int_1^0 (6y^5 - 22y^3 + 4y)\,dy \quad (\because \text{Along } AO \; y \text{ varies from } 1 \text{ to } 0)$$

$$= \left[6\frac{y^6}{6} - \frac{22y^4}{4} + \frac{4y^2}{2} \right]_1^0$$

$$= \frac{5}{2}$$

$$\boxed{\int_{AO} M\,dx + N\,dy = \frac{5}{2}} \longrightarrow (2)$$

$$\therefore \int_C M\,dx + N\,dy = \int_{OA} M\,dx + N\,dy + \int_{AO} M\,dx + N\,dy$$

$$= -1 + \frac{5}{2}$$

$$= \frac{-2+5}{2}$$

$$\boxed{\int_C M\,dx + N\,dy = \frac{3}{2}} \rightarrow (2)$$

$$(1) = (2)$$

Hence Green's theorem is Verified.

Gauss Divergence theorem:

If $\vec{F}$ is a Vector Point function, finite and differentiable in a region R bounded by a closed Surface S, then the surface integral of the normal component of $\vec{F}$ taken over S is equal to the integral of divergence of $\vec{F}$ taken over V.

$$\boxed{\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv}$$

where $\hat{n}$ is the unit Vector in the Positive (outward drawn) normal to S.

Problems:

1) Verify Gauss divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube $x=0, x=1, y=0, y=1, z=0, z=1$.