Stoke's theorem:

If $\vec{F}$ is any continuous differentiable vector function and S is a surface enclosed by a curve c then,

$$\boxed{\int_c \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds}$$

where $\hat{n}$ is the unit normal vector at any point of S.

Note:

1. If $\vec{F}$ is irrotational, $\nabla\times\vec{F}=0$

$$\int_c \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds = 0$$ and hence $\vec{F}$ is conservative.

2. Let $\vec{F} = P\vec{i}+Q\vec{j}+R\vec{k}$

$$\int_c P\,dx + Q\,dy + R\,dz = \iint_S \left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)dydz + \left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)dzdx + \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$$

Problems:

1) Verify Stoke's theorem for a vector field defined by $\vec{F} = (x^2-y^2)\vec{i}+2xy\vec{j}$ in the rectangular region in the xoy plane bounded by the lines $x=0, x=a, y=0$ and $y=b$.

Sol: By Stoke's theorem,

$$\int_c \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds$$

RHS:-

Given: $\vec{F} = (x^2-y^2)\vec{i}+2xy\vec{j}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2 & 2xy & 0 \end{vmatrix}$$

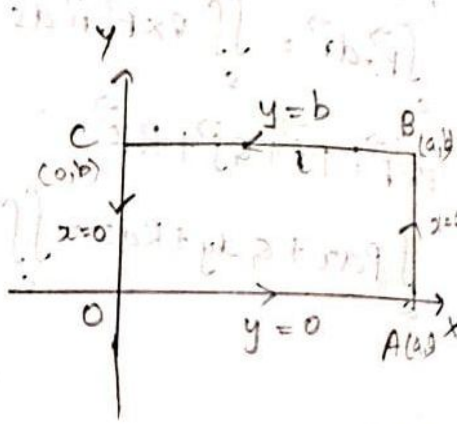
$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(2y+2y)$$

$$= 4y\vec{k}$$

$$\boxed{\nabla \times \vec{F} = 4y\vec{k}}$$

Here the Surface S denotes the rectangle OABC and the unit outward normal vector is $\vec{k}$.

ie.,) $\hat{n} = \vec{k}$



$$\text{Curl } \vec{F}.\hat{n}\,ds = 4y\vec{k}.\vec{k}\,dx\,dy$$

$$= 4y\,dx\,dy$$

$$\therefore \iint \nabla \times \vec{F}.\hat{n}\,ds = \iint_S 4y\,dx\,dy$$

$$= \int_0^b \int_0^a 4y\,dx\,dy$$

$$= 4\int_0^b [x]_0^a\, y\,dy$$

$$= 4a\left[\frac{y^2}{2}\right]_0^b$$

$$= \frac{4ab^2}{2}$$

$$= 2ab^2 \rightarrow (1)$$

$$\boxed{\iint \nabla \times \vec{F}.\hat{n}\,ds = 2ab^2}$$

L.H.S

$$\vec{F} = (x^2-y^2)\vec{i} + 2xy\vec{j}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$\vec{F}\cdot d\vec{r} = (x^2-y^2)\,dx + 2xy\,dy$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \left[(x^2-y^2)\,dx + 2xy\,dy\right]$$

$$= \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

<u>Along OA (y=0):</u>

$$\int_{OA} (x^2-y^2)\,dx + 2xy\,dy = \int_0^a x^2\,dx \qquad \left[\because y=0 \Rightarrow dy=0 \quad \text{Along OA, } x \to 0 \text{ to } a\right]$$

$$= \left[\frac{x^3}{3}\right]_0^a$$

$$= \frac{a^3}{3}$$

$$\boxed{\int_{OA} (x^2-y^2)\,dx + 2xy\,dy = \frac{a^3}{3}}$$

<u>Along AB (x=a):</u>

$$\int_{AB} (x^2-y^2)\,dx + 2xy\,dy = \int_0^b 2ay\,dy \qquad \left[\because x=a \Rightarrow dx=0 \quad \text{Along AB, } y \to 0 \text{ to } b\right]$$

$$= 2a\left[\frac{y^2}{2}\right]_0^b$$

$$= 2a\left[\frac{b^2}{2}\right]$$

$$\boxed{\int_{AB} (x^2-y^2)\,dx + 2xy\,dy = ab^2}$$

Along BC $(y=b)$:

$$\int_{BC} (x^2-y^2)dx + 2xy\,dy = \int_a^0 (x^2-b^2)dx$$

$$= \left[\frac{x^3}{3} - b^2x\right]_a^0$$

$\because [y=b \Rightarrow dy=0$
Along BC $x \to a$ to $0]$

$$= -\frac{a^3}{3} + ab^2$$

$$\boxed{\int_{BC} (x^2-y^2)dx + 2xy\,dy = -\frac{a^3}{3} + ab^2}$$

Along CO $(x=0)$:

$$\int_{CO} (x^2-y^2)dx + 2xy\,dy = \int_{CO} 0+0 = 0 \quad (\because x=0,\ dx=0)$$

$$\boxed{\int_{CO} (x^2-y^2)dx + 2xy\,dy = 0}$$

Hence $\int_C \vec{F}\cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$

$$= \cancel{\frac{a^3}{3}} + ab^2 - \cancel{\frac{a^3}{3}} + ab^2 + 0$$

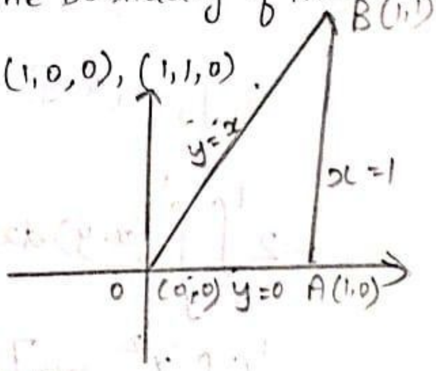
$$= ab^2 + ab^2$$

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = 2ab^2} \longrightarrow (2)$$

From (1) & (2), $\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds}$

Hence Stoke's theorem is verified.

2) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ by Stoke's theorem where $\vec{F} = y^2\vec{i} + x^2\vec{j} - (x+z)\vec{k}$ and c is the boundary of the triangle with vertices at $(0,0,0)$, $(1,0,0)$, $(1,1,0)$.



Sol: By Stoke's theorem

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\, ds$$

Since z coordinate is zero in all the three vertices of the given triangle, the triangle lies on the xy plane.

$$\vec{F} = y^2\vec{i} + x^2\vec{j} - (x+z)\vec{k}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(-1-0) + \vec{k}(2x-2y)$$

$$= \vec{j} + 2(x-y)\vec{k}$$

$$\boxed{\nabla\times\vec{F} = \vec{j} + 2(x-y)\vec{k}}$$

Since the triangle lies on xoy plane and hence the unit vector normal to the surface OAB is $\vec{k}$.

ie,) $\hat{n} = \vec{k}$

$$\nabla\times\vec{F}\cdot\hat{n} = [\vec{j} + 2(x-y)\vec{k}]\cdot\vec{k} = 2(x-y)$$

$$\therefore \int_C \vec{F}\cdot d\hat{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\, ds$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds$$

$$= \iint_S 2(x-y)\,dx\,dy \qquad (\because S \text{ lies on } xy \text{ plane } ds = dx\,dy)$$

$$= 2\int_0^1\int_y^1 (x-y)\,dx\,dy$$

$$= 2\int_0^1 \left[\frac{x^2}{2} - xy\right]_y^1 dy$$

$$= 2\int_0^1 \left[\frac{1}{2} - y - \frac{y^2}{2} + y^2\right] dy$$

$$= 2\left[\frac{1}{2}y - \frac{y^2}{2} - \frac{y^3}{6} + \frac{y^3}{3}\right]_0^1$$

$$= 2\left(\frac{1}{2} - \frac{1}{2} - \frac{1}{6} + \frac{1}{3}\right)$$

$$= 2\left(-\frac{1}{6} + \frac{1}{3}\right)$$

$$= 2\left(\frac{-1+2}{6}\right)$$

$$= 2\left(\frac{1}{6}\right)$$

$$= \frac{1}{3}$$

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \frac{1}{3}}$$