Consider,

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = f(x) \longrightarrow (1)$$

where $a_0, a_1, a_2, \ldots a_n$ are constants and $f(x)$ is a function of x is called a homogeneous linear differential equation of order n with Variable coefficients.

For this type, substitute

$z = \log x$ (or) $x = e^z$

Then $x \frac{dy}{dx} = D'y$

$x^2 \frac{d^2y}{dx^2} = D'(D'-1)y$

$x^3 \frac{d^3y}{dx^3} = D'(D'-1)(D'-2)y$ and so on

Problems :

(1) Transform the equation to constant coefficient equations :

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = \log x \sin(\log x)$$

Soln :

Replace $\log x = z$

$$x \frac{dy}{dx} = D'y$$

$$x^2 \frac{d^2y}{dx^2} = D'(D'-1)y$$

Given equation becomes,

$$D'(D'-1)y + D'y + y = z \sin z$$

$$(D'^2 - D')y + D'y + y = z \sin z$$

$$(D'^2 - D' + D' + 1)y = z \sin z$$

$$\boxed{(D'^2+1)y = z \sin z}$$