Legendre's linear differential equation

$$K_0 (ax+b)^n \frac{d^n y}{dx^n} + K_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + K_n y = f(x)$$

Replace $z = \log(ax+b)$

(or) $e^z = ax+b$

$$(ax+b)\frac{dy}{dx} = a \cdot D' y$$

$$(ax+b)^2 \frac{d^2y}{dx^2} = a^2 \cdot D'(D'-1) y$$

$$(ax+b)^3 \frac{d^3y}{dx^3} = a^3 \cdot D'(D'-1)(D'-2) y \text{ and so on.}$$

Problems:

1. Transform the equation to constant coefficients equation

$$(2x+3)^2 y'' - (2x+3) y' + 2y = 6x$$

Soln:

Put $z = \log(2x+3)$

$e^z = 2x+3$

$\Rightarrow e^z - 3 = 2x \Rightarrow \boxed{x = \frac{e^z - 3}{2}}$

$$(2x+3)\frac{dy}{dx} = 2 \cdot D' y$$

$$(2x+3)^2 \frac{d^2y}{dx^2} = 2^2 \cdot D'(D'-1) y$$

Given equation will be,

$$2^2 D'(D'-1)y - 2D'y + 2y = 6\left(\frac{e^z-3}{2}\right)$$

$$4(D'^2 - D')y - 2D'y + 2y = 3(e^z-3)$$

$$[4D'^2 - 4D' - 2D' + 2]y = 3(e^z-3)$$

$$\boxed{(4D'^2 - 6D' + 2)y = 3(e^z-3)}$$

(2) Transform,

$$(x+2)^2 \frac{d^2y}{dx^2} - (x+2)\frac{dy}{dx} + y = 3x+4 \quad \to (1)$$

Soln:

Put $z = \log(x+2)$

$$e^z = x+2 \Rightarrow x = e^z - 2$$

$$(x+2)\frac{dy}{dx} = D'y$$

$$(x+2)^2 \frac{d^2y}{dx^2} = D'(D'-1)y$$

(1) will becomes,

$$D'(D'-1)y - D'y + y = 3(e^z-2) + 4$$

$$(D'^2 - D' - D' + 1)y = 3(e^z-2) + 4$$

$$(D'^2 - 2D' + 1)y = 3e^z - 6 + 4$$

$$\boxed{(D'^2 - 2D' + 1)y = 3e^z - 2}$$