



DEPARTMENT OF MATHEMATICS

Construction of Analytic function:

Milne-Thomson method:

Let $f(z) = u + iv$ is to be constructed

(i) Suppose the real part u is given, then

$$f(z) = \int [\varphi_1(z,0) - i\varphi_2(z,0)] dz$$

where $\varphi_1(z,0) = \frac{\partial u}{\partial x}(z,0)$, $\varphi_2(z,0) = \frac{\partial u}{\partial y}(z,0)$.

(ii) Suppose imaginary part v is given, then

$$f(z) = \int [\varphi_1(z,0) + i\varphi_2(z,0)] dz$$

where $\varphi_1(z,0) = \frac{\partial v}{\partial y}(z,0)$, $\varphi_2(z,0) = \frac{\partial v}{\partial x}(z,0)$



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- ① Show that the fn $u = x^3 + x^2 - 3xy^2 + 2xy - y^2$ is harmonic and find the corresponding analytic fn $f(z) = u + iv$.

Soln:

$$u = x^3 + x^2 - 3xy^2 + 2xy - y^2$$

$$u_x = 3x^2 + 2x - 3y^2 + 2y \quad ; \quad u_y = -6xy + 2x - 2y$$

$$u_{xx} = 6x + 2 \quad ; \quad u_{yy} = -6x - 2$$

$$u_{xx} + u_{yy} = 0$$

$$\Rightarrow u \text{ is harmonic. } f(z) = \int \left[\frac{\partial u}{\partial x}(z,0) - i \frac{\partial u}{\partial y}(z,0) \right] dz$$

$$f(z) = \int [\varphi_1(z,0) - i \varphi_2(z,0)] dz$$

$$\text{where } \varphi_1(z,0) = \frac{\partial u}{\partial x}(z,0) = 3z^2 + 2z$$

$$\varphi_2(z,0) = \frac{\partial u}{\partial y}(z,0) = 2z$$

$$f(z) = \int [(3z^2 + 2z) - i \cdot 2z] dz$$

$$= \frac{3z^3}{3} + \frac{2z^2}{2} - i \cdot \frac{2z^2}{2} + c$$

$$f(z) = z^3 + z^2(1-i) + c$$

- ② Find an analytic fn whose imaginary part is

$$v = e^{2x} (y \cos 2y + x \sin 2y)$$

Soln:

$$v = e^{2x} (y \cos 2y + x \sin 2y)$$

$$v_x = 2e^{2x} (y \cos 2y + x \sin 2y) + e^{2x} \sin 2y$$

$$v_y = e^{2x} (\cos 2y + y(-2 \sin 2y) + 2x \cos 2y)$$

$$f(z) = \int [\varphi_1(z,0) + i \varphi_2(z,0)] dz$$

$$= \int \left[\frac{\partial v}{\partial y}(z,0) + i \frac{\partial v}{\partial x}(z,0) \right] dz$$



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(3) prove that $u = x^2 - y^2$, $v = \frac{-y}{x^2 + y^2}$ are harmonic

but $u + iv$ is not a regular function.

Soln:

$$\text{Let } u = x^2 - y^2$$

$$\frac{\partial u}{\partial x} = 2x \quad ; \quad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \quad ; \quad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$

$\Rightarrow u$ is a harmonic function.

$$\text{Let } v = \frac{-y}{x^2 + y^2}$$

$$\frac{\partial v}{\partial x} = - \frac{[(x^2 + y^2) \cdot 0 - y(2x)]}{(x^2 + y^2)^2} = \frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2 + y^2)^2 (2y) - 2xy \cdot 2(x^2 + y^2)(2x)}{(x^2 + y^2)^4}$$

$$= \frac{(x^2 + y^2)^2 2y - 8x^2 y (x^2 + y^2)}{(x^2 + y^2)^4}$$

$$= \frac{2y(x^2 + y^2) - 8x^2 y}{(x^2 + y^2)^3}$$

$$= \frac{2y^3 - 6x^2 y}{(x^2 + y^2)^3}$$

$$= \frac{2y^3 - 6x^2 y}{(x^2 + y^2)^3}$$

$$\frac{\partial v}{\partial y} = - \frac{[(x^2 + y^2) - y \cdot 2y]}{(x^2 + y^2)^2} = \frac{-(x^2 - y^2)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2 + y^2)^2 2y - (y^2 - x^2) 2(x^2 + y^2) 2y}{(x^2 + y^2)^4}$$



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$$\begin{aligned}\phi_1(z,0) &= \frac{\partial v}{\partial y}(z,0) = e^{2z}(1+0+2z) \\ &= e^{2z} + 2ze^{2z}\end{aligned}$$

$$\phi_2(z,0) = \frac{\partial v}{\partial x}(z,0) = 2e^{2z}(0) + e^{2z}(0) = 0$$

$$f(z) = \int [e^{2z} + 2ze^{2z}] dz + c$$

$$= \int e^{2z} dz + 2 \int ze^{2z} dz + c$$

$$= \frac{e^{2z}}{2} + 2 \left(\frac{ze^{2z}}{2} - \frac{e^{2z}}{4} \right) + c$$

$$= \frac{e^{2z}}{2} + ze^{2z} - \frac{e^{2z}}{2} + c$$

$$= ze^{2z} + c$$

③ Given that $u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$, find the

analytic f_n whose real part is u .

Soln:

$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$u_x = \frac{(\cosh 2y - \cos 2x) \cos 2x \cdot 2 - \sin 2x (2 \sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$u_y = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x (2 \sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$f(z) = \int [\phi_1(z,0) + \phi_2(z,0)] dz$$

$$\phi_1(z,0) = \frac{\partial u}{\partial x}(z,0)$$

$$= \frac{(\cos 0 - \cos 2z) 2 \cos 2z - \sin 2z (2 \sin 2z)}{(\cos 0 - \cos 2z)^2}$$



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$$= \frac{2 \cos 2z - (2 \cos^2 2z - 2 \sin^2 2z)}{(1 - \cos 2z)^2}$$

$$= \frac{2 \cos 2z - 2}{(1 - \cos 2z)^2}$$

$$= \frac{-2(1 - \cos 2z)}{(1 - \cos 2z)^2}$$

$$= \frac{-2}{1 - \cos 2z} = \frac{-2}{2 \sin^2 z} = -\operatorname{cosec}^2 z$$

$$\begin{aligned} \phi_2(z, 0) &= \frac{(\cos 0 - \cos 2z)(0) - \sin 2z(2 \sin 0)}{(\cos 0 - \cos 2z)^2} \\ &= 0. \end{aligned}$$

$$f(z) = \int -\operatorname{cosec}^2 z \, dz = \cot z + C$$

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