Necessary Condition (Cartesian Coordinates):

(or) Cauchy-Riemann equations:

If the function $f(z) = u(x,y) + iv(x,y)$ is analytic in a region R of the z plane, then

(i) $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ exists

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

at every point in that region.

Sufficient Conditions:

If the function $f(z) = u(x,y) + iv(x,y)$ is analytic in a region R of the z-plane if

(i) u_x, u_y, v_x & v_y are exists and all are continuous.

(ii) $u_x = v_y$ and $u_y = -v_x$.

Necessary Condition (polar Coordinates):

If the function $w = f(z) = u(r,\theta) + iv(r,\theta)$ is analytic in a region R of the z-plane then

(i) if $\frac{\partial u}{\partial r}$, $\frac{\partial u}{\partial \theta}$, $\frac{\partial v}{\partial r}$ and $\frac{\partial v}{\partial \theta}$ exists.

(ii) $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

Sufficient Conditions:

If the function $w = f(z) = u(r,\theta) + iv(r,\theta)$ is analytic in a region R of the z-plane, then

(i) $\frac{\partial u}{\partial r}$, $\frac{\partial u}{\partial \theta}$, $\frac{\partial v}{\partial r}$ and $\frac{\partial v}{\partial \theta}$ exists & all are Continuous

(ii) $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

PROBLEMS:

(1) Prove that $\omega = z^2$ is analytic.

Soln:
We know $z = x + iy$

$\therefore \omega = z^2$
$= (x+iy)^2$
$= x^2 - y^2 + 2ixy$

$u + iv = (x^2 - y^2) + i(2xy)$

$u = x^2 - y^2 \quad ; \quad v = 2xy$

$u_x = 2x \qquad v_x = 2y$

$u_y = -2y \qquad v_y = 2x$

$u_x = v_y \;\&\; u_y = -v_x$

It satisfies CR equations.

$\Rightarrow \omega = z^2$ is analytic.

(2) Determine whether the function $\omega = 2xy + i(x^2 - y^2)$ is analytic.

Soln: $\omega = 2xy + i(x^2 - y^2)$

$u + iv = 2xy + i(x^2 - y^2)$

$u = 2xy \;, \quad v = x^2 - y^2$

$u_x = 2y \;, \quad v_x = 2x$

$u_y = 2x \qquad v_y = -2y$

$u_x \neq v_y$

$\&\; u_y \neq -v_x$

It doesn't satisfies CR equations

$\Rightarrow \omega = 2xy + i(x^2 - y^2)$ is not analytic

(3) Verify whether $f(z) = \sinh z$ is analytic using CR eqns.

Soln:

$f(z) = \sinh z$

$\sin i\theta = i\sinh\theta$

$\cos i\theta = \cosh\theta$

$u + iv = \sinh(x+iy)$

$= \frac{1}{i}\sin i(x+iy)$ (Multiply & divide by i)

$= \frac{1}{i}\sin(ix + i^2 y)$

$= \frac{1}{i}\sin(ix - y)$

$= \frac{1}{i}[\sin ix\cos y - \cos ix\sin y]$

$= \frac{1}{i}[i\sinh x\cos y - \cosh x\sin y]$

$= \sinh x\cos y - \frac{1}{i}\cosh x\sin y$

$= \sinh x\cos y + i\cosh x\sin y \quad \left[\frac{1}{i} = -i\right]$

$u = \sinh x\cos y, \quad v = \cosh x\sin y$

$u_x = \cosh x\cos y, \quad v_x = \sinh x\sin y$

$u_y = -\sinh x\sin y, \quad v_y = \cosh x\cos y$

$u_x = v_y \;\&\; u_y = -v_x$

$\therefore$ It satisfies CR equations.

$\therefore f(z) = \sinh z$ is analytic.

(4) Show that $f(z) = |z|^2$ is nowhere analytic.

Soln:

$f(z) = |z|^2$

$u + iv = x^2 + y^2$

$u = x^2 + y^2, \quad v = 0$

$u_x = 2x$, $v_x = 0$

$u_y = 2y$, $v_y = 0$

$\Rightarrow u_x \neq v_y$ & $u_y \neq -v_x$

It doesn't satisfies CR eqns.

$\therefore f(z) = |z|^2$ is nowhere analytic.

(5) If $u+iv$ is analytic then $v-iu$ is also analytic.

Soln:

$u+iv$ is analytic

i.e., CR eqns are satisfied.

i.e., $u_x = v_y$ & $u_y = -v_x$.

i.e., $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

To prove: $v-iu$ is also analytic.

i.e., We have to prove, $\frac{\partial v}{\partial x} = \frac{\partial(-u)}{\partial y}$ & $\frac{\partial v}{\partial y} = -\frac{\partial(-u)}{\partial x}$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \& \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

We know that,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

Hence $v-iu$ is also analytic.

(6) If $w = e^z$, find $\frac{dw}{dz}$ using complex variable.

Soln:

$w = e^z$

$u+iv = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$

$= e^x \cos y + i\, e^x \sin y$

$u = e^x \cos y$, $v = e^x \sin y$

$u_x = e^x \cos y$, $v_x = e^x \sin y$

Result : If $w = f(z) = u+iv$ then

$$\frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

finding $\frac{dw}{dz}$ in terms of partial derivatives w.r.t x)

$$\Rightarrow \frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= e^x \cos y + i e^x \sin y$$

$$= e^x(\cos y + i \sin y)$$

$$= e^x e^{iy}$$

$$= e^{x+iy}$$

$$= e^z.$$