Conformal mapping :

(1) Find the image of the following region under the translation $w = 1/z$

(i) half plane $x > c$ when $c > 0$

(ii) the infinite strip $\frac{1}{4} < y < 1/2$

(iii) the infinite strip $0 < y < 1/2$.

Soln:

$$w = \frac{1}{z}$$

$$z = \frac{1}{w}$$

$$x + iy = \frac{1}{u+iv} = \frac{1}{u+iv} \cdot \frac{u-iv}{u-iv}$$

$$x + iy = \frac{u-iv}{u^2+v^2} = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$$

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2}$$

(i) Half plane $x > c$ when $c > 0$

$$x = c$$

$$\frac{u}{u^2+v^2} = c$$

$$u = c(u^2+v^2)$$

$$\frac{u}{c} = u^2+v^2$$

$$u^2 - \frac{u}{c} + v^2 = 0.$$

$$\left(u^2 - \frac{u}{c} + \left(\frac{1}{2c}\right)^2\right) + v^2 - \left(\frac{1}{2c}\right)^2 = 0$$

$$\left(u - \frac{1}{2c}\right)^2 + v^2 = \left(\frac{1}{2c}\right)^2$$

which is a circle with centre $\left(\frac{1}{2c}, 0\right)$ & radius $1/2c$.

$u^2 - \frac{u}{c}$

$a = u$

$2ab = \frac{u}{c}$

$b = \frac{u}{2ac}$

$b = \frac{\cancel{u}}{2\cancel{u}c}$

$b = \frac{1}{2c}$

(ii) the infinite strip $\frac{1}{4} < y < \frac{1}{2}$

$y = \frac{1}{4}$

$$\frac{-v}{u^2+v^2} = \frac{1}{4}$$

$$-v = \frac{1}{4}(u^2+v^2)$$

$$-4v = u^2+v^2$$

$$u^2+v^2+4v = 0$$

$$u^2+(v+2)^2-4 = 0$$

$$u^2+(v+2)^2 = 4$$

which is a eqn of circle with centre $(0,-2)$ & $r = 2$

$y = \frac{1}{2}$

$$\frac{-v}{u^2+v^2} = \frac{1}{2}$$

$$-2v = u^2+v^2$$

$$u^2+v^2+2v = 0$$

$$u^2+(v+1)^2-1 = 0$$

$$u^2+(v+1)^2 = 1$$

centre: $(0,-1)$

$r = 1$

(iii) $0 < y < 1/2$

$y = 0$

$$\frac{-v}{u^2+v^2} = 0$$

$$\boxed{v = 0}$$

which is a straight line in w-plane.

$y = \frac{1}{2}$

$$\frac{-v}{u^2+v^2} = \frac{1}{2}$$

$$-2v = u^2+v^2$$

$$u^2+v^2+2v = 0$$

$$u^2+(v+1)^2-1 = 0$$

$$u^2+(v+1)^2 = 1$$

centre : $(0,-1)$

radius $= 1$