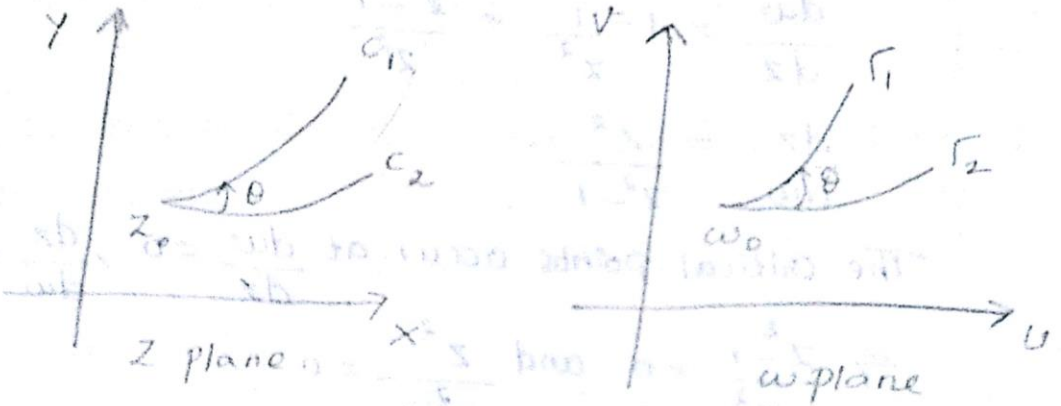
Conformal mapping :

Defn:

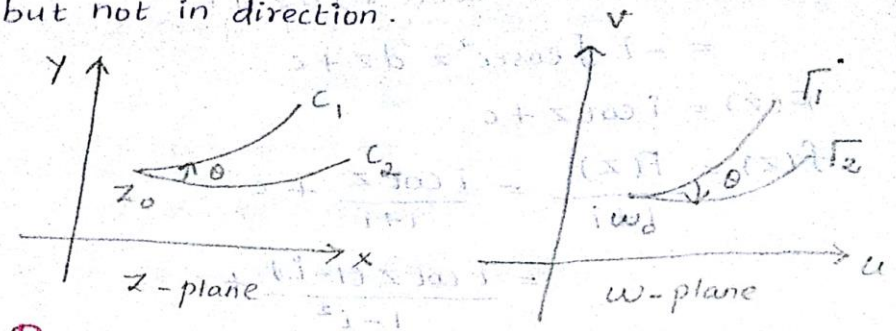
A mapping $w = f(z)$ is said to be Conformal at $z = z_0$ if it preserves the angle between any two curves through z_0 in z plane both in magnitude and direction.



Isogonal mapping :

A mapping $w = f(z)$ is said to be isogonal at $z = z_0$ if it preserves the angle between any two curves through z_0 in z plane only in magnitude

but not in direction.



Remarks :

1. If $f(z)$ is analytic and $f'(z) \neq 0$ at each Point then the mapping $w = f(z)$ is conformal.
2. The points at which $w = f(z)$ is not conformal i.e., $f'(z) = 0$ are called Critical points.
3. The Critical Points of $w = f(z)$ will occur at $\frac{dz}{dw} = 0$ and $\frac{dw}{dz} = 0$.

(1) Find the Critical points of the transformation $w = z + 1/z$.

Soln:

$$\frac{dw}{dz} = 1 - \frac{1}{z^2} = \frac{z^2-1}{z^2}$$

$$\frac{dz}{dw} = \frac{z^2}{z^2-1}$$

The Critical points occur at $\frac{dw}{dz} = 0$, $\frac{dz}{dw} = 0$.

$\Rightarrow \frac{z^2-1}{z^2} = 0$ and $\frac{z^2}{z^2-1} = 0$

$\Rightarrow z^2 - 1 = 0$ and $z^2 = 0$

$\Rightarrow z = \pm 1$ and $z = 0$

$\therefore z = 0, 1, -1$ are the Critical points.

(2) Find the critical points of $w^2 = (z-\alpha)(z-\beta)$.

Soln:

$$w^2 = (z-\alpha)(z-\beta)$$

$$2w\frac{dw}{dz} = (z-\alpha)+(z-\beta)$$

$$\frac{dw}{dz} = \frac{(z-\alpha)+(z-\beta)}{2w}$$

$$\frac{dz}{dw} = \frac{2w}{(z-\alpha)+(z-\beta)}$$

$$\frac{dw}{dz} = \frac{(z-\alpha)+(z-\beta)}{2w} \quad \& \quad \frac{dz}{dw} = \frac{2w}{(z-\alpha)+(z-\beta)}$$

∴ The critical points occur at $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$.

$$\Rightarrow \frac{z-\alpha+z-\beta}{2w} = 0 \quad \text{and} \quad \frac{2w}{(z-\alpha)+(z-\beta)} = 0$$

$$\Rightarrow 2z = \alpha+\beta \qquad w = 0$$

$$\Rightarrow z = \frac{\alpha+\beta}{2} \qquad \Rightarrow w^2 = 0$$

$$\Rightarrow (z-\alpha)(z-\beta) = 0$$

$$\Rightarrow z = \alpha, \beta$$

∴ $z = \alpha, \beta, \frac{\alpha+\beta}{2}$ are the critical points.

(3) Find the points such that $w = f(z) = \sin z$ is not conformal.

Soln: Let $w = \sin z$

$$\frac{dw}{dz} = \cos z \; ; \; \frac{dz}{dw} = \frac{1}{\cos z}$$

The critical points occur at $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$.

$\Rightarrow \cos z = 0$ and $1/\cos z = 0$

$\Rightarrow z = \cos^{-1}(0)$ $\quad 1 = 0$, impossible

$z = \pm (2n-1)\frac{\pi}{2}$, $n = 0, 1, 2, \ldots$