Unit - II

Ordinary Differential Equations

Second Order linear differential equation with Constant coefficients :.

Consider a Second order linear differential equation is $(a_0D^2 + a_1D + a_2)y = R(x)$

To find Complementary function :.

The auxiliary equation is $a_0m^2 + a_1m + a_2 = 0$.

Nature of Roots	Complementary function (C.F)
1. m_1 and m_2 are real and different.	$C.F = Ae^{m_1x} + Be^{m_2x}$
2. m_1 and m_2 are real and equal.	$C.F = (Ax+B)e^{mx}$
3. m_1 and m_2 are complex. Let $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$	$C.F = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$.

Problems to find Complementary function :

1) Solve : $(D^2 - 5D + 6)y = 0$

Sol: The Auxiliary equation is

$$m^2 - 5m + 6 = 0$$

$$m = 2, 3 \quad (m_1, m_2)$$

The roots are real and different.

$$C.F = Ae^{m_1 x} + Be^{m_2 x}$$

$$= Ae^{2x} + Be^{3x}$$

$$C.F = Ae^{2x} + Be^{3x}$$

2) Solve : $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$.

Sol: Given : $(D^2 - 6D + 9)y = 0$

The Auxiliary equation is

$$m^2 - 6m + 9 = 0$$

$$m = 3, 3$$

The roots are real and equal.

$$C.F = (Ax + B)e^{mx}$$

$$= (Ax + B)e^{3x}$$

$$y = (Ax + B)e^{3x}$$

Rules to find Particular integral (P.I):

$$\text{Particular Integral} = \frac{1}{f(D)} R(x)$$

Failure Case: when $f(D) = 0$

$$P.I = \frac{x}{f'(D)} R(x)$$

Again failure, when $f'(D) = 0$

$$P.I = \frac{x^2}{f''(D)} R(x)$$

and so on.

Type I: $R(x) = e^{ax}$ Replace $D \to a$.

1) Solve: $(D^2+1)y = e^{-x}$

Sol:- The auxiliary equation is

$$m^2+1 = 0$$

$$m^2 = -1$$

$$m = \pm i = 0 \pm i$$

$\alpha = 0, \beta = 1$

$$C.F = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

$$= e^{0x}(A\cos x + B\sin x)$$

$$C.F. = A\cos x + B\sin x$$

$P.I = \frac{e^{-x}}{D^2+1} = \frac{e^{-x}}{(-1)^2+1}$

$e^{ax} = e^{-x}$

Here $a = -1$

$D \to a \to -1$

$= \frac{e^{-x}}{2}$

$y = C.F + P.I$

$= A\cos x + B\sin x + \frac{e^{-x}}{2}$

$y = A\cos x + B\sin x + \frac{e^{-x}}{2}$

2) Solve: $(D^2+4D+4)y = 11e^{-2x}$

Sol:- The auxiliary equation is

$m^2+4m+4 = 0$

$(m+2)^2 = 0$

$m = -2, -2$

The roots are real and equal.

$C.F = (Ax+B)e^{mx}$

$C.F = (Ax+B)e^{-2x}$

$e^{ax} = e^{-2x}$

$a = -2$

$D \to a \to -2$

$P.I = \frac{11e^{-2x}}{D^2+4D+4}$

$= \frac{11e^{-2x}}{(-2)^2+4(-2)+4}$

$$= \frac{11e^{-2x}}{4-8+4}$$

$$= \frac{11e^{-2x}}{0}$$ (failure case)

$$= \frac{11x e^{-2x}}{2D+4}$$

$$= \frac{11x e^{-2x}}{2(-2)+4} = \frac{11x e^{-2x}}{-4+4} = \frac{11x e^{-2x}}{0}$$ (failure)

$$P.I = \frac{11x^2 e^{-2x}}{2}$$

$y = C.F + P.I$

$$y = (Ax+B)e^{-2x} + \frac{11x^2 e^{-2x}}{2}$$

3) Solve : $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 4$

Sol: $(D^2+4D+3)y = 4$

The Auxiliary equation is

$m^2+4m+3=0$

$m = -1, -3$

$C.F = Ae^{-x} + Be^{-3x}$

$$P.I = \frac{4}{D^2+4D+3}$$

$= 4 \cdot \frac{e^{0x}}{D^2+4D+3}$

$= \frac{4e^{0x}}{3}$

$e^{0x} = e^{ax}$

$a = 0$

$D \to a \to 0$

$P.I = \frac{4}{3}$

$y = C.F + P.I$

$= Ae^{-x} + Be^{-3x} + \frac{4}{3}$

$y = Ae^{-x} + Be^{-3x} + \frac{4}{3}$

Type: II $R(x) = \sin ax$ (or) $\cos ax$

Replace $D^2 \to -a^2$

1) Solve: $(D^2+4)y = \sin 3x$

Sol:- The Auxiliary equation is

$m^2+4 = 0$

$m^2 = -4$

$m = \pm 2i = 0 \pm 2i$

$\alpha = 0, \beta = 2$

$C.F = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$= e^{0x}(A\cos 2x + B\sin 2x)$

$C.F = A\cos 2x + B\sin 2x$

$P.I = \frac{\sin 3x}{D^2+4}$

$\sin ax = \sin 3x$

$a = 3$

$D^2 \to -a^2 \to -3^2 = -9$

$= \frac{\sin 3x}{-9+4}$

$= \frac{\sin 3x}{-5}$

$$P.I = \frac{\sin 3x}{-5}$$

Here, the Solution is

$$y = C.F + P.I$$

$$y = A\cos 2x + B\sin 2x - \frac{\sin 3x}{5}$$

2) Find the P.I of $(D^2+1)^2 y = \sin 2x$

Sol:- $P.I = \frac{\sin 2x}{(D^2+1)^2}$

$\sin ax = \sin 2x$

$a = 2$

$D^2 \to -a^2 \to -2^2 = -4$

$= \frac{\sin 2x}{(-4+1)^2}$

$$P.I = \frac{\sin 2x}{9}$$

3) Find the P.I of $(D^2+4)y = \cos 2x$

Sol:- $P.I = \frac{\cos 2x}{D^2+4}$

$\cos ax = \cos 2x$

$a = 2$

$D^2 \to -a^2 \to -2^2$

$= -4$

$= \frac{\cos 2x}{-4+4}$

$= \frac{\cos 2x}{0}$ (failure)

$= \frac{x\cos 2x}{2D}$

$= \frac{x}{2}\cdot\frac{1}{D}(\cos 2x)$

$= \frac{x}{2}\cdot\frac{\sin 2x}{2}$

$= \frac{x\sin 2x}{4}$

1) Solve: $(D^2+3D+2)y = e^{-3x}$

Sol:- The auxiliary equation is $m^2+3m+2=0$

$m = -1, -2$

The roots are real and different.

$C.F = Ae^{-x} + Be^{-2x}$

$P.I = \frac{1}{D^2+3D+2} e^{-3x}$

Sub $D=-3 \Rightarrow \frac{1}{9-3(3)+2} e^{-3x}$

$= \frac{1}{9-9+2} e^{-3x}$

$= \frac{1}{2} e^{-3x}$

$y = C.F + P.I$

$y = Ae^{-x} + Be^{-2x} + \frac{1}{2} e^{-3x}$

2) Solve: $(D^2+4D+4)y = 11e^{-2x}$

Sol:- The auxiliary equation is $m^2+4m+4=0$

$m = -2, -2.$

The roots are real and equal.

$C.F = (A+Bx)e^{mx}$

$= (A+Bx)e^{-2x}$

$P.I = \frac{1}{D^2+4D+4} 11e^{-2x}$

Sub $D=-2$

$$= \frac{1}{4+4(-2)+4} 11e^{-2x}$$

$$= \frac{1}{4-8+4} 11e^{-2x}$$

$$= \frac{1}{0} 11e^{-2x}$$

$$= \frac{x}{2D+4} 11e^{-2x}$$

$$= \frac{x}{-4+4} 11e^{-2x}$$

$$= \frac{x}{0} 11e^{-2x}$$

$$= \frac{x^2}{2} 11e^{-2x}$$

$$P.I = \frac{x^2}{2} 11e^{-2x}$$

$$y = C.F + P.I$$

$$y = (A+Bx)e^{-2x} + \frac{x^2}{2} 11e^{-2x}$$

Type : 2

R.H.S $\rightarrow \sin(ax+b)$ or $\cos(ax+b)$

1) Solve : $(D^2-3D+2)y = \sin 3x$.

Sol :- The auxiliary equation is

$$m^2-3m+2=0$$

$$m=1,2$$

The roots are real and different.

$C.F = Ae^{x} + Be^{2x}$

$P.I = \frac{1}{D^2-3D+2} \sin 3x$

$D^2 \to -(a^2) = -(3^2) = -9$

$= \frac{1}{-9-3D+2} \sin 3x$

$= \frac{1}{-7-3D} \sin 3x$

$= \frac{1}{-3D-7} \sin 3x$

$= \frac{1}{-3D-7} \times \frac{-3D+7}{-3D+7} \sin 3x$

$= \frac{-3D+7}{9D^2-49} \sin 3x$

$D^2 = -9$ $= \frac{-3D+7}{-81-49} \sin 3x$

$= \frac{-3D+7}{-130} \sin 3x$

$= \frac{-1}{130}\left[-3D(\sin 3x) + 7(\sin 3x)\right]$

$= \frac{-1}{130}\left[-3\cos 3x(3) + 7\sin 3x\right]$

$= \frac{-1}{130}\left[-9\cos 3x + 7\sin 3x\right]$

$y = C.F + P.I$

$y = Ae^{x} + Be^{2x} + \left(\frac{-1}{130}\right)\left[-9\cos 3x + 7\sin 3x\right]$

2) $(D^2-3D+2)y = 2\cos(2x+3)+2e^x$

Sol:- The auxiliary equation is $m^2-3m+2=0$

$m=1,2$

The roots are real and different.

$C.F = Ae^x+Be^{2x}$

$P.I_1 = \frac{1}{D^2-3D+2}\,2\cos(2x+3)$

$D^2 \to (a^2)$

$-(2^2)=-4$

$= 2\,\frac{1}{-4-3D+2}\cos(2x+3)$

$= 2\,\frac{1}{-3D-2}\cos(2x+3)$

$= 2\left[\frac{1}{-3D-2}\times\frac{-3D+2}{-3D+2}\right]\cos(2x+3)$

$= 2\left[\frac{-3D+2}{9D^2-4}\right]\cos(2x+3)$

$= 2\left[\frac{-3D+2}{9(-4)-4}\right]\cos(2x+3)$

$= 2\left[\frac{-3D+2}{-36-4}\right]\cos(2x+3)$

$= 2\left[\frac{-3D+2}{-40}\right]\cos(2x+3)$

$P.I_1 = \frac{-1}{20}\left[-3D\cos(2x+3)+2\cos(2x+3)\right]$