Binomial Expansion:

1) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

2) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

3) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

4) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

Method of Variation of Parameters:

The Second order linear differential equation is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R.$$

Step 1: Find Complementary function. From this calculate y_1 and y_2 (coefficient of constant).

Step 2: Find Wronskian

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

where

$$P = -\int \frac{R}{W} y_2 \, dx$$

$$Q = \int \frac{R}{W} y_1 \, dx$$

$$P.I = Py_1 + Qy_2$$

1) $(D^2+4)y = \sec 2x$.

Sol:: The auxiliary equation is $m^2+4=0$

$$m^2=-4$$

$$m=\pm 2i$$

The roots are imaginary.

$$C.F = e^{0x}\left[A\cos 2x + B\sin 2x\right]$$

$f_1 = \cos 2x$ $\qquad$ $f_2 = \sin 2x$

$f_1' = -2\sin 2x$ $\qquad$ $f_2' = 2\cos 2x$

$$w = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos^2 2x + 2\sin^2 2x$$

$$= 2(\cos^2 2x + \sin^2 2x)$$

$$w = 2$$

$$P.I = Pf_1 + Qf_2$$

$P = -\int \frac{f_2 X}{W}\,dx$ $\qquad$ $Q = \int \frac{f_1 X}{W}\,dx$

$= -\int \frac{\sin 2x \sec 2x}{2}\,dx$ $\qquad$ $= \int \frac{\cos 2x \sec 2x}{2}\,dx$

$= -\frac{1}{2}\int \sin 2x \cdot \frac{1}{\cos 2x}\,dx$ $\qquad$ $= \frac{1}{2}\int \cos 2x \cdot \frac{1}{\cos 2x}\,dx$

$= \frac{-1}{2}\int \tan 2x \, dx$

$= \frac{-1}{2} \frac{\log(\sec 2x)}{2}$

$= \frac{-1}{4} \log \sec 2x$

$= \frac{1}{2}\int dx$

$= \frac{1}{2}(x)$

$= \frac{x}{2}$

$P.I = Pf_1 + Qf_2$

$= -\frac{1}{4}\log(\sec 2x)\cos 2x + \frac{x}{2}\sin 2x.$

$y = C.F + P.I$

$= A\cos 2x + B\sin 2x - \frac{1}{4}\log(\sec 2x)\cos 2x + \frac{x}{2}\sin 2x$

2) Solve: $(D^2+1)y = \operatorname{cosec} x.$

The auxiliary equation is $m^2+1=0$

$m = \pm i$

The roots are imaginary.

$C.F = e^{0x}[A\cos x + B\sin x]$

$= A\cos x + B\sin x$

$f_1 = \cos x \qquad f_2 = \sin x$

$f_1' = -\sin x \qquad f_2' = \cos x$

$W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x$$

$$= 1$$

$$P.I = Pf_1 + Qf_2$$

$$P = -\int \frac{f_2 X}{W} dx$$

$$= -\int \frac{\sin x \operatorname{cosec} x}{1} dx$$

$$= -\int \sin x \cdot \frac{1}{\sin x} dx$$

$$= -\int dx$$

$$= -x$$

$$Q = \int \frac{f_1 X}{W} dx$$

$$= \int \frac{\cos x \operatorname{cosec} x}{1} dx$$

$$= \int \cos x \cdot \frac{1}{\sin x} dx$$

$$= \int \cot x \, dx$$

$$= \log \sin x$$

$$P.I = -x\cos x + \log(\sin x)\sin x$$

$$y = C.F + P.I$$

$$= -A\cos x + B\sin x - x\cos x + \sin x \log \sin x$$

$$\boxed{y = -A\cos x + B\sin x - x\cos x + \log(\sin x)\sin x}$$

3) $(D^2+1)y = \cot x$

Sol: The auxiliary equation is $m^2+1=0$

$m^2=-1$

$m=\pm i$

The roots are imaginary.

$C.F = e^{0x}[A\cos x + B\sin x]$

$= A\cos x + B\sin x$

$f_1 = \cos x \qquad f_2 = \sin x$

$f_1' = -\sin x \qquad f_2' = \cos x$

$W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$

$= \cos^2 x + \sin^2 x$

$= 1$

$P.I = Pf_1 + Qf_2$

$P = -\int \frac{f_2 X}{W} dx$

$= -\int \frac{\sin x \cot x}{1} dx$

$= -\int \sin x \times \frac{\cos x}{\sin x} dx$

$= -\int \cos x \, dx$

$= -\sin x$

$P = -\sin x$

$Q = \int \frac{f_1 X}{W} dx$

$= \int \frac{\cos x \cot x}{1} dx$

$= \int \frac{\cos^2 x}{\sin x} dx$

$= \int \frac{1-\sin^2 x}{\sin x} dx$

$= \int \left(\frac{1}{\sin x} - \frac{\sin^2 x}{\sin x}\right) dx$

$= \int \operatorname{cosec} x \, dx - \int \sin x \, dx$

$= -\log(\operatorname{cosec} x + \cot x) - (-\cos x)$

$$P.I = -\sin x \cos x + [-\log(\operatorname{cosec} x + \cot x) + \cos x]\sin x$$

$$y = C.F + P.I$$

$$y = A\cos x + B\sin x - \sin x\cos x + [-\log(\operatorname{cosec} x + \cot x) + \cos x]\sin x.$$

4) Solve: $(D^2+4)y = \tan 2x$.

Sol:- The auxiliary equation is

$$m^2+4=0$$

$$m^2=-4$$

$$m=\pm 2i$$

$$C.F = e^{0x}[A\cos 2x + B\sin 2x]$$

$$= A\cos 2x + B\sin 2x$$

$$f_1 = \cos 2x \qquad f_2 = -\sin x$$

$$f_1' = -2\sin 2x \qquad f_2' = 2\cos 2x$$

$$W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos^2 2x + 2\sin^2 2x$$

$$= 2(\cos^2 2x + \sin^2 2x)$$

$$= 2(1)$$

$$W = 2$$

$$P.I = Pf_1 + Qf_2$$

$$P = -\int \frac{f_2 X}{W} dx \qquad Q = \int \frac{f_1 X}{W} dx$$

$$= -\int \frac{\sin 2x \tan 2x}{2} dx \qquad = \int \frac{\cos 2x \tan 2x}{2} dx$$

$$= \frac{-1}{2} \int \frac{\sin^2 2x}{\cos 2x} dx \qquad = \int \cos 2x \frac{\sin 2x}{\cos 2x} dx$$

$$= \frac{-1}{2} \int \frac{1 - \cos^2 2x}{\cos 2x} dx \qquad = \int \sin 2x \, dx$$

$$= \frac{-1}{2} \int \frac{1}{\cos 2x} dx - \frac{\cos^2 2x}{\cos 2x} dx \qquad = \frac{-\cos 2x}{2}$$

$$= \frac{-1}{2} \int \sec 2x \, dx - \cos 2x \, dx$$

$$= \frac{-1}{2} \left[\left[\frac{\log(\sec 2x + \tan 2x)}{2} \right] - \frac{\sin 2x}{2} \right]$$

$$P.I = Pf_1 + Qf_2$$

$$= \frac{-1}{2} \left[\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2} \right] \cos 2x - \frac{\cos 2x}{2} \sin 2x.$$

$$y = C.F + P.I$$

$$y = A\cos 2x + B\sin 2x - \frac{1}{2}\left[\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2} \right] \cos 2x - \frac{\cos 2x}{2} \sin 2x.$$