Legendre's Linear Differential Equations:

$$(ax+b)^2 y'' + (ax+b)y' + y = Q(x)$$

Let $e^z = (ax+b)$

$z = \log(ax+b)$

$(ax+b)D = a\theta$

$(ax+b)^2 D^2 = a^2(\theta^2 - \theta)$

1. Transform the equation to construct coefficient

$$(2x+3)^2 y'' - (2x+3)y' + 2y = 6x$$

$$(2x+3)^2 D^2 y - (2x+3)Dy + 2y = 6x$$

$$\left[(2x+3)^2 D^2 - (2x+3)D + 2\right] y = 6x \longrightarrow (1)$$

$e^z = 2x+3$

$z = \log(2x+3)$

$2x = e^z - 3$

$x = \frac{e^z - 3}{2}$

$(2x+3)D = 2\theta$

$(2x+3)^2 D^2 = 2^2(\theta^2 - \theta)$

$= 4(\theta^2 - \theta)$

Sub in equ (1)

$$\left(4(\theta^2-\theta) - 2\theta + 2\right) y = 6\left(\frac{e^z - 3}{2}\right)$$

$$(4\theta^2 - 4\theta - 2\theta + 2)y = 3e^z - 9$$

$$(4\theta^2 - 6\theta + 2)y = 3e^z - 9$$

2) Solve : $[(x+2)^2D^2-(x+2)D+1]y = 3x+4.$

Sol:- $[(x+2)^2D^2+(x+2)D+1]y = 3x+4 \rightarrow (1)$

$e^z = x+2$

$x = e^z - 2$

$z = \log(x+2)$

$(x+2)D = \theta$

$(x+2)D^2 = 1^2(\theta^2-\theta) = \theta^2-\theta$

Sub in equ (1)

$[(\theta^2-\theta)-\theta+1]y = 3[e^z-2]+4$

$(\theta^2-\theta-\theta+1)y = 3e^z-6+4$

$(\theta^2-\theta(2)+1)y = 3e^z-2 \rightarrow (2)$

C.F The auxiliary equation is $m^2-2m+1=0$

$(m-1)(m-1)=0$

$m=1,1$

$C.F = (A+Bz)e^z$

$P.I_1 = \frac{1}{\theta^2-2\theta+1}3e^z$

$= \frac{1}{1^2-2(1)+1}3e^z$

$= \frac{1}{0}3e^z$

$P.I_1 = \frac{z}{2\theta-2}3e^z = \frac{z}{2(1)-2}3e^z = \frac{z}{0}3e^z = \frac{z}{2}3e^z$

$$PI_2 = \frac{1}{D^2-2D+1}(-2e^{0z})$$

$$= \frac{1}{0^2-2(0)+1}(-2(1))$$

$$= \frac{1}{1}(-2)$$

$$= -2$$

$$y = C.F + PI_1 + PI_2$$

$$= (A+Bz)e^z + \frac{z}{2}3e^z - 2$$

$$= [A+B\log(x+2)]\, e^{\log(x+2)} + \frac{3\log(x+2)^2}{2} e^{\log(x+2)} - 2$$

$$= [A+B\log(x+2)(x+2) + \frac{3\log(x+2)^2}{2}(x+2) - 2].$$

$$\boxed{y = [A+B\log(x+2)(x+2) + \frac{3\log(x+2)^2}{2}(x+2) - 2]}$$