Unit - II

Ordinary Differential Equations

Second order linear differential equation with constant coefficients:

Consider a second order linear differential equation is $(a_0D^2 + a_1D + a_2)y = R(x)$

To find Complementary function:

The auxiliary equation is $a_0m^2 + a_1m + a_2 = 0$.

Nature of roots	Complementary function (C.F)
1. m_1 and m_2 are real and different.	$C.F = Ae^{m_1 x} + Be^{m_2 x}$
2. m_1 and m_2 are real and equal.	$C.F = (Ax+B)e^{mx}$
3. m_1 and m_2 are complex. Let $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$	$C.F = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$.

Problems to find Complementary function:

1) Solve: $(D^2-5D+6)y=0$

Sol: The Auxiliary equation is

$$m^2-5m+6=0$$

$$\boxed{m=2,3} \quad (m_1, m_2)$$

The roots are real and different.

$$C.F = Ae^{m_1 x}+Be^{m_2 x}$$

$$= Ae^{2x}+Be^{3x}$$

$$\boxed{C.F = Ae^{2x}+Be^{3x}}$$

2) Solve: $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=0$.

Sol: Given: $(D^2-6D+9)y=0$

The Auxiliary equation is

$$m^2-6m+9=0$$

$$\boxed{m=3,3}$$

The roots are real and equal.

$$C.F = (Ax+B)e^{mx}$$

$$= (Ax+B)e^{3x}$$

$$\boxed{y=(Ax+B)e^{3x}}$$

Rules to find Particular integral (P.I):

$$\text{Particular Integral} = \frac{1}{f(D)} R(x)$$

Failure case: when $f(D) = 0$

$$P.I = \frac{x}{f'(D)} R(x)$$

Again failure, when $f'(D) = 0$

$$P.I = \frac{x^2}{f''(D)} R(x)$$

and so on.

Type I : $R(x) = e^{ax}$ Replace $D \to a$.

1) Solve: $(D^2+1)y = e^{-x}$

Sol: The auxiliary equation is

$$m^2+1 = 0$$

$$m^2 = -1$$

$$m = \pm i = 0 \pm i$$

$\alpha = 0, \ \beta = 1$

$$C.F = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

$$= e^{0x}(A\cos x + B\sin x)$$

$$C.F. = A\cos x + B\sin x$$

$$P.I = \frac{e^{-x}}{D^2+1} = \frac{e^{-x}}{(-1)^2+1}$$

$e^{ax} = e^{-x}$

Here $a = -1$

$D \to a \to -1$

$$= \frac{e^{-x}}{2}$$

$y = C.F + P.I$

$= A\cos x + B\sin x + \frac{e^{-x}}{2}$

$$y = A\cos x + B\sin x + \frac{e^{-x}}{2}$$

2) Solve: $(D^2+4D+4)y = 11e^{-2x}$

Sol:- The auxiliary equation is

$m^2+4m+4 = 0$

$(m+2)^2 = 0$

$m = -2, -2$

The roots are real and equal.

$C.F = (Ax+B)e^{mx}$

$$C.F = (Ax+B)e^{-2x}$$

$e^{ax} = e^{-2x}$

$a = -2$

$D \to a \to -2$

$$P.I = \frac{11e^{-2x}}{D^2+4D+4}$$

$$= \frac{11e^{-2x}}{(-2)^2+4(-2)+4}$$

$$= \frac{11e^{-2x}}{4-8+4}$$

$$= \frac{11e^{-2x}}{0} \quad \text{(failure case)}$$

$$= \frac{11xe^{-2x}}{2D+4}$$

$$= \frac{11xe^{-2x}}{2(-2)+4} = \frac{11xe^{-2x}}{-4+4} = \frac{11xe^{-2x}}{0} \quad \text{(failure)}$$

$$\boxed{P.I = \frac{11x^2e^{-2x}}{2}}$$

$$y = C.F + P.I$$

$$\boxed{y = (Ax+B)e^{-2x} + \frac{11x^2e^{-2x}}{2}}$$

3) Solve : $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 4$

Sol:- $(D^2+4D+3)y = 4$

The Auxiliary equation is

$$m^2+4m+3=0$$

$$\boxed{m=-1,-3}$$

$$C.F = Ae^{-x} + Be^{-3x}$$

$$P.I = \frac{4}{D^2+4D+3}$$

$$= 4 \cdot \frac{e^{0x}}{D^2+4D+3}$$

$$e^{0x} = e^{ax}$$

$$a = 0$$

$$D \to a \to 0$$

$$= \frac{4e^{0x}}{3}$$

$$\boxed{P.I = \frac{4}{3}}$$

$$y = C.F + P.I$$

$$= Ae^{-x} + Be^{-3x} + \frac{4}{3}$$

$$\boxed{y = Ae^{-x} + Be^{-3x} + \frac{4}{3}}$$