DEPARTMENT OF AEROSPACE ENGINEERING

Faculty Name	:	**Dr.A.Arun Negemiya, ASP/ Aero**	Academic Year	:	**2024-2025 (Even)**
Year & Branch	:	**III AEROSPACE**	Semester	:	**VI**
Course	:	**19ASB304 - Computational Fluid Dynamics for Aerospace Application**			

UNIT III – FINITE ELEMENT TECHNIQUES

Weighted Residual and Galerkin Formulations

Weighted Residual Formulation

The first basic ingredient of the finite element method is that an approximate solution is sought which belongs to some finite dimension function space. This function space is to be specified more in detail later. For the time being, we look for an approximate solution to the boundary value problem (3.1, 3.2, and 3.3) which has the form

$$\hat{u} = \psi + \sum_{k=1}^{N} \phi_k \, u_k \qquad (3.14)$$

where ψ is a function which satisfies the boundary conditions (3.2) and (3.3). For the given problem, the construction of ψ is obvious. The functions θk are called basis functions or shape functions. Since the dimension of the function space Φ = {θk; k = 1, 2, . . . , N} is finite, in general, an expression of type (3.14) cannot satisfy the differential equation (3.1) in each point of the domain. This means that the approximate solution ˆu cannot be identical with the exact solution u. Of course, the shape functions should be chosen so that by enriching the function space Φ, i.e. letting N grow, the approximation obtained by (10.14) becomes better. This means that the approximate solution converges to the exact solution. This is called the completeness requirement of the function space. Since a function ˆu given by (3.14) cannot satisfy the differential equation (3.1), upon substitution of (3.14) into (3.1), a residual is left:

$$r_\Omega = a(\hat{u}) - f \quad \text{in } \Omega \qquad (3.15)$$

An approximate solution to the boundary value problem now is obtained by finding a way to make this residual small in some sense. In the finite element method this is done by requiring that an appropriate number of weighted integrals of the residual over Ω be zero:

$$\int_{\Omega} w_i r_\Omega d\Omega = 0; \quad i = 1, 2, \ldots, N \qquad (3.16)$$

where W= {wi; i = 1, 2, . . . , N} is a set of weighting functions. The convergence requirement now also implies a requirement of completeness of the space of weighting functions, i.e. (3.16) should imply $r\Omega \to 0$ for $N \to \infty$.

Clearly, with the satisfaction of the completeness, for $N \to \infty$, the weighted residual formulation (3.16) for a function of the form (3.14) is completely equivalent to the strong formulation of the problem (3.1, 3.2 and 3.3). An approximate solution then is obtained for N being finite.

Galerkin Formulation

Among the possible choices for the set of weighting functions, the following ones are the most obvious.

The weighting functions can be chosen to be Dirac-delta functions in N points. This choice means making the residual equal to zero in some chosen points. The method is called the point collocation method. It has much in common with the finite difference methodology. A second possible choice of weighting functions is given by

$$\begin{aligned} w_i &= 1 \quad \text{for } x_i \le x \le x_{i+1} \\ &= 0 \quad \text{for } x < x_i \text{ or } x > x_{i+1} \end{aligned}$$

The weighted residual statements (3.16) now require the integral of the residual to be zero on N subdomains. This method is called the subdomain collocation method. The finite volume method, in which not the differential form of the equation but the integral form of the equation is discretized, is a special form of this method.

The most popular choice for the weighting functions in the finite element method is the shape functions themselves:

$$w_i = \phi_i$$

This method is called the Galerkin method. Its meaning is that the residual is made to be orthogonal to the space of the shape functions.

To illustrate the Galerkin method, consider the boundary value problem (3.1–3.3) with constant λ. Then:

$$\psi = u_0 + \frac{q}{\lambda} x$$

Consider further as an example of (3.14) a Fourier-sine expansion of u:

$$\hat{u} = \psi + \sum_{k=1}^{N} u_k \sin \frac{\pi k' x}{X}, \quad \text{with } k' = k - 1/2$$

Then:

$$r_\Omega = -\lambda \sum_{k=1}^{N} u_k \left(\frac{\pi k'}{X}\right)^2 \sin \frac{\pi k' x}{X} - f$$

The Galerkin method then gives

$$\lambda \sum_{k=1}^{N} u_k \left(\frac{\pi k'}{X}\right)^2 \int_0^X \sin \frac{\pi k' x}{X} \sin \frac{\pi i' x}{X} dx = -\int_0^X \sin \frac{\pi i' x}{X} f\, dx$$

Then noting that

$$\begin{aligned}\int_0^X \sin \frac{\pi k' x}{X} \sin \frac{\pi i' x}{X} dx &= \frac{X}{2} \quad \text{for } k' = i' \\ &= 0 \quad \text{for } k' \neq i'\end{aligned}$$

we find

$$u_i = -\frac{2X}{\lambda \pi^2 i'^2} \int_0^X f \sin \frac{\pi i' x}{X} dx$$

The foregoing method used to determine an approximate solution of the boundary value problem (3.1, 3.2, and 3.3) is not a finite element method, but a spectral method. The finite element method however has the same starting point.

Before going on with the study of the building blocks of the finite element method, we should remark that a fourth weighted residual statement exists on which finite element methods can be based. The least squares formulation is based on the minimization of the integral.

$$\int_\Omega r_\Omega^2 d\Omega$$