

UNIT – II, TWO DIMENSIONAL RANDOM VARIABLES

PART-A

1. The bivariate random variable X and Y has the pdf

$$f(x,y)=\begin{cases} kx^2(8-y), & x < y < 2x \\ 0 & \text{elsewhere} \end{cases} \quad \text{find } k.$$

Ans:

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dy dx &= 1 \\ \int_0^2 \int_x^{2x} kx^2(8-y) dy dx &= 1 \quad k \int_0^2 x^2 \left(8y - \frac{y^2}{2} \right) dx = 1 \\ k \int_0^2 x^2 \left(16x - \frac{4x^2}{2} - 8x + \frac{x^2}{2} \right) dx &= 1 \quad k \int_0^2 \left(16x^3 - 2x^4 - 8x^3 + \frac{x^4}{2} \right) dx = 1 \\ k \int_0^2 \left(8x^3 - \frac{3x^4}{2} \right) dx &= 1 \quad k \left[\frac{8x^4}{4} - \frac{3x^5}{10} \right]_0^2 = 1 \\ k \left[32 - \frac{48}{5} \right] &= 1 \quad k \left[\frac{112}{5} \right] = 1 \\ k \left[\frac{112}{5} \right] &= 1 \\ k &= \frac{5}{112} \end{aligned}$$

2. The joint pdf of random variable x and y is given by $f(x,y) = kxye^{-(x^2+y^2)}$, $x > 0, y > 0$ find the value of k.

Ans:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dx dy = 1$$

$$\int_0^{\infty} \int_0^{\infty} kxye^{-(x^2+y^2)} dydx = 1$$

$$k \int_0^{\infty} ye^{-y^2} dy \int_0^{\infty} xe^{-x^2} dx = 1 \quad \left[\int_0^{\infty} xe^{-x^2} dx = \frac{1}{2} \right]$$

$$k \frac{1}{2} \cdot \frac{1}{2} = 1, \quad k = 4$$

3. If X and Y have joint pdf $f(x,y) = \begin{cases} x+y, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$. check whether X and Y are independent.

Ans:

The marginal density of X is

$$f(x) = \int_{-\infty}^{\infty} f(x,y) dy \quad f(x) = \int_0^1 (x+y) dy$$

$$f(x) = \left[xy + \frac{y^2}{2} \right]_0^1 \quad f(x) = x + \frac{1}{2}$$

$$f(y) = \int_{-\infty}^{\infty} f(x,y) dx \quad f(y) = \int_0^1 (x+y) dx$$

$$f(y) = \left[\frac{x^2}{2} + xy \right]_0^1 \quad f(y) = \frac{1}{2} + y$$

$$f(x).f(y) = \left(x + \frac{1}{2}\right) \left(y + \frac{1}{2}\right) \quad f(x).f(y) \neq f(x,y)$$

4. Let X and Y have j.d.f $f(x,y)=2, 0 < x < y < 1$. Find m.d.f

Ans:

Marginal density of X is given by

$$\begin{aligned}
 f(x) &= \int_{-\infty}^{\infty} f(x,y)dy = \int_x^1 2dy \\
 &= 2[y]_x^1 \\
 &= 2(1-x), 0 < x < 1.
 \end{aligned}$$

Marginal density function of Y is given by

$$f(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^y 2dx = 2[x]_0^y = 2y, 0 < y < 1.$$

5. The j.d.f of the random variables X and Y is given by

$$f(x,y) = \begin{cases} 8xy, & 0 < x < 1, 0 < y < x \\ 0, & \text{otherwise} \end{cases} \text{ find } f_x(x).$$

Ans:

$$\begin{aligned}
 f_x(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\
 &= \int_0^x 8xydy = 8x \left(\frac{y^2}{2} \right)_0^x \\
 &= 8x \left(\frac{x^2}{2} \right)
 \end{aligned}$$

$$f_x(x) = 4x^3, 0 < x < 1$$

6. Given $f(x,y) = \begin{cases} cx(x-y), & 0 < x < 2, -x < y < x \\ 0, & \text{otherwise} \end{cases}$, find c.

Ans:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)dydx = 1$$

$$\int_0^2 \int_{-x}^x cx(x-y)dydx=1$$

$$c \int_0^2 \left[x^2y - x \cdot \frac{y^2}{2} \right]_{-x}^x dx=1$$

$$c \int_0^2 \left[x^3 - \frac{x^2}{2} + x^3 + \frac{x^3}{2} \right] dx=1$$

$$c \int_0^2 2x^3 dx=1$$

$$2c \left[\frac{x^4}{4} \right]_0^2 = 1 \qquad 2c \left[\frac{16}{4} \right] = 1 \qquad c = \frac{1}{8}$$

6. The joint p.d.f of a bivariate random variable (X,Y) is given by

$$f(x,y) = \begin{cases} kxy, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}, \text{ find K.}$$

Ans:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dx dy = 1$$

$$\int_0^1 \int_0^1 kxy dx dy = 1 \qquad k \int_0^1 \left[\frac{x^2}{2} y \right]_0^1 dy = 1$$

$$k \int_0^1 \frac{y}{2} dy = 1 \qquad k \left[\frac{y^2}{4} \right]_0^1 = 1$$

$$k = 4$$

7. If the joint pdf of (x,y) is $f(x,y) = \frac{1}{4}, 0 < x, y < 1$, find $p(x+y \leq 1)$.

Ans:

$$p(x + y \leq 1) = p(x \leq 1 - y)$$

$$= \int_0^1 \int_0^{1-y} f(x, y) dx dy$$

$$= \int_0^1 \int_0^{1-y} \frac{1}{4} dx dy = \frac{1}{4} \int_0^1 [x]_0^{1-y} dy$$

$$= \frac{1}{4} \int_0^1 [(1-y)] dy = \frac{1}{4} \left[y - \frac{y^2}{2} \right]_0^1$$

$$= \frac{1}{4} \left[1 - \frac{1}{2} \right] = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$$

8. Two random variables X and Y have joint pdf $f(x, y) = \begin{cases} \frac{xy}{96}, & 0 < x < 4, 1 < y < 5 \\ 0, & \text{otherwise} \end{cases}$, find E(x).

Ans:

$$E(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xf(x, y) dx dy$$

$$= \int_1^5 \int_0^4 x \left(\frac{xy}{96} \right) dx dy = \frac{1}{96} \int_1^5 \left[y \cdot \frac{x^2}{2} \right]_0^4 dy$$

$$= \frac{1}{96} \int_1^5 \frac{64}{2} y dy = \frac{64}{288} \left[\frac{y^2}{2} \right]_1^5$$

$$= \frac{2}{9} \left[\frac{25}{2} - \frac{1}{2} \right] = \frac{1}{9} (24)$$

$$= \frac{8}{3}$$

9. Let X be a Random variable with pdf $f(x) = \frac{1}{2}, -1 \leq x \leq 1$, and let $Y = X^2$, find $E(Y)$.

Ans:

$$\begin{aligned}
 Y &= x^2 \\
 E(Y) &= E(x^2) \\
 \int_{-\infty}^{\infty} x^2 f(x) dx &= \int_{-1}^1 x^2 \left(\frac{1}{2}\right) dx = \frac{1}{2} \left(\frac{x^3}{3}\right)_{-1}^1 = \frac{1}{6}(2) = \frac{1}{3}.
 \end{aligned}$$

10. If the joint pdf of (x,y) is given by $f(x,y) = x+y, 0 \leq x, y \leq 1$. Find $E(XY)$.

Ans:

$$\begin{aligned}
 E(XY) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x,y) dx dy \\
 &= \int_0^1 \int_0^1 xy(x+y) dx dy \\
 &= \int_0^1 \int_0^1 (x^2 y + xy^2) dx dy \\
 &= \int_0^1 \left(\frac{x^3}{3} y + \frac{x^2}{2} y^2 \right)_{x=0}^1 dy \\
 &= \int_0^1 \left(\frac{y}{3} + \frac{y^2}{2} \right) dy \\
 &= \left(\frac{y^2}{6} + \frac{y^3}{6} \right)_{y=0}^1 = \frac{2}{6} = \frac{1}{3}.
 \end{aligned}$$

11. Find the acute angle between the two lines of regression

Ans:

The acute angle between the two lines of regression is $\tan \theta = \frac{1-r^2}{r} \left(\frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2} \right)$.

12. State the equation of the two regression lines. What is the formula for correlation coefficient

Ans:

X on Y is $(x - \bar{x}) = b_{xy}(y - \bar{y})$ and Y on X is $(y - \bar{y}) = b_{yx}(x - \bar{x})$.

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Correlation coefficient $r = \sqrt{b_{xy} \cdot b_{yx}}$.

13. If X and Y are independent random variables with variance 2 and 3. Find the variance of 3X+4Y.

Ans:

$$\text{Var}(x) = 2, \text{Var}(y) = 3$$

$$\begin{aligned}\text{Var}(3X + 4Y) &= 3^2 \text{Var}(X) + 4^2 \text{Var}(Y) \\ &= 9\text{Var}(X) + 16\text{Var}(Y) \\ &= 9 \cdot 2 + 16 \cdot 3 \\ &= 66\end{aligned}$$

14. The joint pdf of (X,Y) is given by $e^{-(x+y)}, 0 < x, y < \infty$. Are X and Y independent?

Ans :

$$\begin{aligned}f(x) &= \int_{-\infty}^{\infty} f(x, y) dy \\ &= \int_0^{\infty} e^{-x} e^{-y} dy \\ &= e^{-x} \left(-e^{-y} \right)_0^{\infty} = -e^{-x} (0 - 1) = e^{-x}.\end{aligned}$$

$$\begin{aligned}f(y) &= \int_{-\infty}^{\infty} f(x, y) dx \\ &= \int_0^{\infty} e^{-x} e^{-y} dx \\ &= e^{-y} \left(-e^{-x} \right)_0^{\infty} = -e^{-y} (0 - 1) = e^{-y}.\end{aligned}$$

$$f(x)f(y) = e^{-x} e^{-y} = e^{-(x+y)} = f(x, y) \quad \text{Therefore, X and Y are independent.}$$

15. The two lines of regression are $8x - 10y + 66 = 0$, $40x - 18y - 214 = 0$. Find the mean value of X and Y.

Ans:

$$8x - 10y = -66 \quad (1)$$

$$40x - 18y = 214 \quad (2)$$

Solving (1) and (2), we get $x = 104, y = 17$

Mean of X = 13

Mean of Y = 17.

16. The two regression lines are $x = \frac{9}{20}y + \frac{107}{20}$, $y = \frac{4}{5}x + \frac{33}{5}$. Find correlation coefficient?

Ans:

$$r = \sqrt{b_{xy} \cdot b_{yx}}$$

$$\text{Here, } b_{xy} = \frac{9}{20}, b_{yx} = \frac{4}{5}$$

$$r = \sqrt{\frac{9}{20} \times \frac{4}{5}} = 0.6$$

17. If the pdf of X is $f(x) = 2x$, $0 < x < 1$. Find the pdf of $y = 3x + 1$.

Ans:

$$\text{Given } y = 3x + 1$$

$$\frac{dy}{dx} = 3 \Rightarrow \frac{dx}{dy} = \frac{1}{3}$$

$$f_y(y) = f(x) \left| \frac{dx}{dy} \right| \qquad Y = 3X + 1,$$

$$= 2x \left(\frac{1}{3} \right) \qquad 3X = Y - 1$$

$$= 2 \left(\frac{y-1}{3} \right) \left(\frac{1}{3} \right) \qquad X = \frac{Y-1}{3}$$

$$= \frac{2}{9}(y-1).$$

When $x=0$, $y=1$

When $x=1$, $y=4$ $f(y) = \frac{2}{9}(y-1), 1 < y < 4$

18. State Central Limit theorem.

Liapounoff's form:

If $X_i (1, 2, \dots, n)$ be n independent random variable such that $E(X) = \mu$ and $Var(X) = \sigma^2$, then under certain general conditions, the random variable $S_n = X_1 + X_2 + \dots + X_n$ is asymptotically normal with mean μ and standard deviation σ .
