Laurent's Series

Let C_1 and C_2 be two concentric circles $|z-a| = R_1$ and $|z-a| = R_2$ where $R_2 < R_1$.

Let $f(z)$ be analytic inside and on the annular region R between C_1 and C_2. Then for any $z \in R$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

where $a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{(z-a)^{n+1}} dz$

$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{(z-a)^{1-n}} dz$

Problems :

(1) Expand $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ in a Laurent's series if (i) $|z| < 2$ (ii) $|z| > 3$ and (iii) $2 < |z| < 3$.

Soln: Using partial fractions,

$$f(z) = \frac{z^2-1}{(z+2)(z+3)} = A + \frac{B}{z+2} + \frac{C}{z+3} \rightarrow (1)$$

$$\frac{z^2-1}{\cancel{(z+2)}\cancel{(z+3)}} = \frac{A(z+2)(z+3) + B(z+3) + C(z+2)}{\cancel{(z+2)}\cancel{(z+3)}}$$

$$z^2-1 = A(z+2)(z+3) + B(z+3) + C(z+2)$$

Put $z = -2$

$(-2)^2 - 1 = A(0) + B(-2+3) + 0$

$4 - 1 = B$

$\boxed{B = 3}$

Put $z = -3$

$(-3)^2 - 1 = 0 + 0 + C(-3+2)$

$9 - 1 = -C$

$\boxed{C = -8}$

Put $z = 0$

$0 - 1 = A(2)(3) + B(3) + C(2)$

$-1 = 6A + 3B + 2C$

$= 6A + 3(3) + 2(-8)$

$= 6A + 9 - 16$

$-1 = 6A - 7$

$6A = 7 - 1 = 6$

$\boxed{A = 1}$

$(1) \Rightarrow f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3} \rightarrow (2)$

(i) $|z| < 2$

$\Rightarrow \frac{|z|}{2} < 1$

(2) $\Rightarrow f(z) = 1 + \dfrac{3}{2\left(1+\frac{z}{2}\right)} - \dfrac{8}{3\left(1+\frac{z}{3}\right)}$

$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$

$= 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \cdots\right]$

$- \frac{8}{3}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \cdots\right]$

Formula

$\boxed{\left(\because (1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots\right)}$

(ii) $|z| > 3$

$3 < |z|$

$\frac{3}{|z|} < 1$.

(2) $\Rightarrow f(z) = 1 + \dfrac{3}{z\left(1+\frac{2}{z}\right)} - \dfrac{8}{z\left(1+\frac{3}{z}\right)}$

$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$

$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \cdots\right]$

$- \frac{8}{z}\left[1 - \frac{3}{z} + \left(\frac{3}{z}\right)^2 - \cdots\right]$

(iii) $2 < |z| < 3$

$2 < |z|$ and $|z| < 3$

$\frac{2}{|z|} < 1$ and $\frac{|z|}{3} < 1$

$$② \Rightarrow 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \cdots\right]$$

$$- \frac{8}{3}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \cdots\right]$$

Soln:

Given : $f(z) = \dfrac{7z-2}{z(z-2)(z+1)}$

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$$

$$7z-2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$$

When $z=0$, $A=1$

$z=-1$, $C=-3$

$z=2$, $B=2$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

(i) $|z|<2 \Rightarrow \left|\frac{z}{2}\right|<1$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$= \frac{1}{z} + \frac{\cancel{2}}{-\cancel{2}(1-\frac{z}{2})} - \frac{3}{z+1}$$

$$= \frac{1}{z} - \left(1-\frac{z}{2}\right)^{-1} - 3(z+1)^{-1}$$

$$= \frac{1}{z} - \left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{3}\right)^3+\cdots\right] - 3\left[1-z+z^2-z^3+\cdots\right]$$

(ii) $|z|>3 \Rightarrow \frac{3}{|z|}<1$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$= \frac{1}{z} + \frac{2}{z(1-\frac{2}{z})} - \frac{3}{z(1+\frac{1}{z})}$$

$$= \frac{1}{z} + \frac{2}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{3}{z}\left(1+\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z} + \frac{2}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\cdots\right] - \frac{3}{z}\left[1-\frac{1}{z}+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+\cdots\right]$$

(iii) $2 < |z| < 3$

$|z| > 2 \quad , \quad |z| < 3$

$\Rightarrow \dfrac{2}{|z|} < 1 \quad , \quad \dfrac{|z|}{3} < 1$

$f(z) = \dfrac{1}{z} + \dfrac{2}{z-2} - \dfrac{3}{z+1}$

$= \dfrac{1}{z} + \dfrac{2}{z\left(1-\frac{2}{z}\right)} - \dfrac{3}{z\left(1+\frac{1}{z}\right)}$

$= \dfrac{1}{z} + \dfrac{2}{z}\left(1+\dfrac{2}{z}+\left(\dfrac{2}{z}\right)^2+\cdots\right) - \dfrac{3}{z}\left(1-\dfrac{1}{z}+\left(\dfrac{1}{z}\right)^2-\cdots\right]$

(iv) $1 < |z+1| < 3$

Let $u = z+1 \Rightarrow z = u-1$

$1 < |u| < 3$

$\Rightarrow \dfrac{1}{|u|} < 1 \quad , \quad \left|\dfrac{u}{3}\right| < 1$

$f(z) = \dfrac{1}{u-1} + \dfrac{2}{u-3} - \dfrac{3}{u}$

$= \dfrac{1}{u\left[1-\frac{1}{u}\right]} + \dfrac{2}{(-3)\left(1-\frac{u}{3}\right)} - \dfrac{3}{u}$

$= \dfrac{1}{u}\left(1-\dfrac{1}{u}\right)^{-1} - \dfrac{2}{3}\left(1-\dfrac{u}{3}\right)^{-1} - \dfrac{3}{u}$

$= \dfrac{1}{u}\left(1+\dfrac{1}{u}+\left(\dfrac{1}{u}\right)^2+\cdots\right) - \dfrac{2}{3}\left(1+\dfrac{u}{3}+\left(\dfrac{u}{3}\right)^2+\cdots\right)$
$- 3/u$

$f(z) = \dfrac{1}{z+1}\left[1+\left(\dfrac{1}{z+1}\right)+\left(\dfrac{1}{z+1}\right)^2+\cdots\right]$
$-\dfrac{2}{3}\left(1+\dfrac{z+1}{3}+\left(\dfrac{z+1}{3}\right)^2+\cdots\right) - \dfrac{3}{z+1}$