DEPARTMENT OF AEROSPACE ENGINEERING

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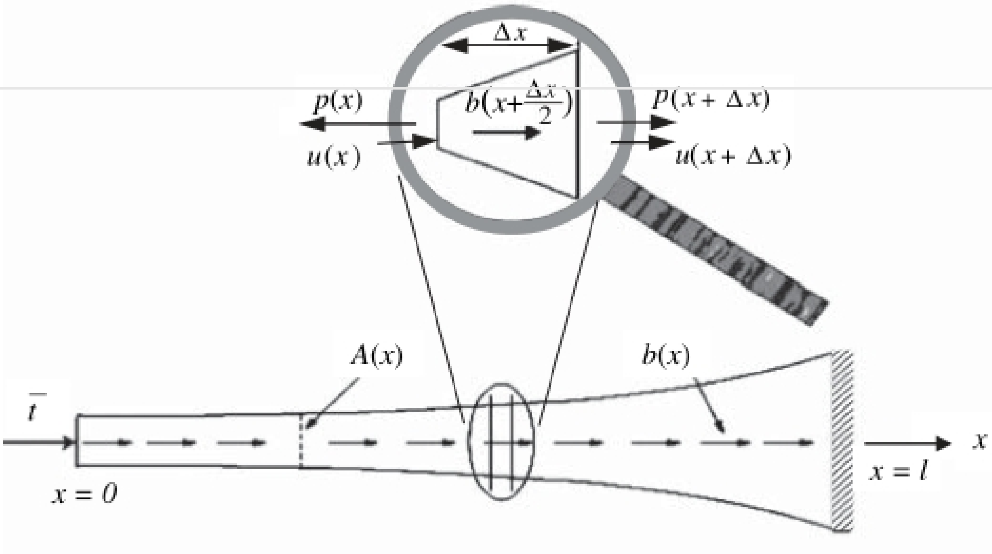
UNIT III – FINITE ELEMENT TECHNIQUES

Strong and Weak Formulations in CFD

The mathematical models of heat conduction and elastostatics consist of (partial) differential equations with initial and boundary conditions. This is also referred to as the so-called **Strong Form** of the problem. In this chapter, we shall therefore discuss the formulation in terms of so-called **strong** and **weak** forms. To facilitate this, we consider simple differential equations, which turn out to govern one-dimensional heat flow as well as other physical phenomena like elastic bars, flexible strings, etc. To obtain a firm background, it is convenient first to establish this differential equation. So let's get started!!!

The partial differential equations in the last paragraph are **second-order partial differential equations**. This demands a high degree of smoothness for the solution u(x). That means that the second derivative of the displacement has to **exist** and has to be **continuous**! This also implies requirements for parameters that are not influenceable like the geometry (sharp edges) and material parameters (different Young's modulus in a material).

The strong form for an axially loaded bar



Consider the static response of an elastic bar with variable cross section as shown in the picture above. This is an example of a problem in linear *stress analysis* or *linear elasticity*, where we seek to find the stress distribution $\sigma(x)$ in the bar. The stress will result from the deformation of the body, which is characterized by the displacement of points in the body, $u(x)$. This displacement implies a strain denoted by $\epsilon(x)$. You can see in the picture that the body is subjected to a body force or distributed loading $b(x)$ (units are force per length). In addition, we can describe the body force which could be due to gravity. Furthermore, loads can be prescribed at the ends of the bar, where the displacement is not prescribed $\rightarrow$ these loads are called **tractions** and denoted by $\bar{t}$ (units are force per area $\rightarrow$ multiplied with an area give us the applied force).

The bar must satisfy the following conditions:

1. Equilibrium must be fulfilled
2. Stress-Strain law must be satisfied. $\sigma(x) = E(x)\epsilon(x)$
3. Displacement field must be compatible
4. Strain-Displacement equations must be satisfied

The differential equation of this bar can be obtained from equilibrium of external forces $b(x)$ as well as the internal forces $p(x)$ acting on the body in the axial direction (along x-axis).

Summing the forces in x-direction:

$$-p(x) + b\left(x + \frac{\Delta x}{2}\right)\Delta x + p(x + \Delta x) = 0$$

Rearranging the terms:

$$\frac{p(x + \Delta x) - p(x)}{\Delta x} + b\left(x + \frac{\Delta x}{2}\right) = 0$$

The limit of this equation with $\Delta x \to 0$ makes the first term the derivative dp/dx and the second term becomes $b(x)$. So we can write:

$$\frac{dp}{dx} + b(x) = 0 \tag{1}$$

This equation expresses the equilibrium equation in terms of the internal force p. Stress is defined as:

$$\sigma(x) = \frac{p(x)}{A(x)} \qquad so \quad p(x) = A(x)\sigma(x) \tag{2}$$

The strain-displacement equation is obtained by:

$$\epsilon(x) = \frac{elongation}{originallength} = \frac{u(x + \Delta x) - u(x)}{\Delta x}$$

Taking the limit of above for $\Delta x \to 0$, we see that:

$$\epsilon(x) = \frac{du}{dx} \tag{3}$$

The stress-strain law, also known as Hooke's law has already been introduced in earlier chapters:

$$\sigma(x) = E(x)\epsilon(x) \tag{4}$$

Substituting (3) in (4) and that result into (1) yields:

$$\frac{d}{dx}\left(AE\frac{du}{dx}\right) + b = 0, \qquad 0 < x < l \tag{5}$$

Equation (5) is a second-order ordinary differential equation. $u(x)$ is the dependent variable, which is the unknown function, and x the independent variable. Equation (5) is a specific form of equation (1). Equation (1) applies both linear and nonlinear materials whereas (5) assumes linearity in the definition of the strain (3) and stress-strain law (4). Compatibility is satisfied by requiring the displacement to be continuous.

To solve the differential equation, we need to prescribe boundary conditions at the two ends of the bar. At $x = l$, the displacement, $u(x = l)$, is prescribed; at $x = 0$, the force per unit area, or traction, denoted by $\bar{t}$, is prescribed. We write these conditions as:

$$\sigma(0) = \left(E\frac{du}{dx}\right)_{x=0} = \frac{p(0)}{A(0)} = -\bar{t} \tag{6}$$
$$u(l) = \overline{u}$$

Note that the lines above the letters indicate a prescribed boundary value.

The traction $\bar{t}$ has the same units as stress (force/area), but its sign is positive when it acts in the positive x-direction regardless of which face it is acting on, whereas the stress is positive in tension and negative in compression, so that on a negative face a positive stress corresponds to a negative traction.

The governing differential equation (5) along with the boundary conditions (6) is called the **strong form** of the problem.

To summarize, the strong form consists of the governing equation and the boundary conditions, which for example are

$$\begin{aligned} &(a) \qquad \frac{d}{dx}\left(AE\frac{du}{dx}\right) + b = 0 \quad on \quad 0 < x < l \\ &(b) \qquad \sigma(x=0) = \left(E\frac{du}{dx}\right)_{x=0} = -\bar{t} \\ &(c) \qquad u(x=l) = \bar{u} \end{aligned} \tag{7}$$

The weak form (1D)

To develop the finite element formulation, the partial differential equations must be restated in an integral form called the **weak form**. The weak form and the strong form are **equivalent**! In stress analysis the weak form is called the **principle of virtual work**.

We start by multiplying the governing equation (7a) and the traction boundary condition (7b) by an **arbitrary** function $w(x)$ and integrating over the domains on which they hold: for the governing equation, the pertinent domain is $[0, l]$. For the traction boundary, it is the cross-sectional area at x = 0 (no integral needed since this condition only holds only at a point but we multiply it with A). The results are:

$$\begin{aligned} &(a) \qquad \int_0^l w\left[\frac{d}{dx}\left(AE\frac{du}{dx}\right) + b\right] dx = 0 \qquad \forall w \\ &(b) \qquad \left(wA\left(E\frac{du}{dx} + \bar{t}\right)\right)_{x=0} = 0 \qquad \forall w \end{aligned} \tag{13}$$

The function w is called **weight function** or **test function**. In the above, $\forall w$ denotes that $w(x)$ is an arbitrary function, i.e. (13) has to hold for all functions $w(x)$. Arbitrariness of the weight function is crucial for the weak form. Otherwise the strong form is **NOT** equivalent to the weak form.

We did not enforce the boundary condition on the displacement in (13) by the weight function. It will be seen that it is easy to construct trial solutions $u(x)$ that satisfy this boundary condition. We will also see that all weight functions satisfy

$$w(l) = 0 \tag{14}$$

In solving the weak form, a set of **admissible solutions** $u(x)$ that satisfy certain conditions is considered (also called **trial solutions** or **candidate solutions**). We could use equation (13) to construct a FEM method. But since we have the second derivative of $u(x)$$ in the equation, we would need very smooth trial functions which are difficult to construct in more than one dimension.
The resulting stiffness matrix would also be not symmetric! We will transform (13) into a form containing only first derivatives. This will give us a symmetric stiffness matrix and allows us to use less smooth solutions.

We rewrite (13a) in an equivalent form:

$$\int_0^l w\frac{d}{dx}\left(AE\frac{du}{dx}\right)dx + \int_0^l wbdx = 0 \qquad \forall w \tag{15}$$

Applying integration by parts on equation (15):

$$\int_0^l w\frac{d}{dx}\left(AE\frac{du}{dx}\right)dx = \left(wAE\frac{du}{dx}\right)\Bigg|_0^l - \int_0^l \frac{dw}{dx}AE\frac{du}{dx}dx \tag{16}$$

Using (16),equation (15) can be written as:

$$\left(wAE\overbrace{\frac{du}{dx}}^{\sigma}\right)\Bigg|_0^l - \int_0^l \frac{dw}{dx}AE\frac{du}{dx}dx + \int_0^l wbdx = 0 \qquad \forall w \quad with \quad w(l) = 0$$

We note that by the stress-strain law and strain-displacement equations, the underscored boundary term is the stress σ:

$$(wA\sigma)_{x=l} - (wA\sigma)_{x=0} - \int_0^l \frac{dw}{dx}AE\frac{du}{dx}dx + \int_0^l wbdx \quad \forall w \quad with \quad w(l) = 0 \tag{18}$$

The first term in equation (18) vanishes since we assumed $w(l) = 0$: for that reason is it useful to construct weight functions that vanish on prescribed displacement boundaries. Though the term looks insignificant, it would lead to loss of symmetry in the final equation. Form (13b), we can see that the second term equals $(wA\bar{t})_{x=0}$, so equation (18) becomes

$$\int_0^l \frac{dw}{dx}AE\frac{du}{dx}dx = (wA\bar{t})_{x=0} + \int_0^l wbdx \quad \forall w \quad with \quad w(l) = 0 \tag{19}$$

To summarize the approach: We multiplied the governing equation and traction boundary by an **arbitrary**, **smooth** weight function and integrated the products over the domain where they hold. We also transformed the integral so that the derivatives are of lower order.

The crux of this approach: Trial solutions that satisfy the equation developed for all smooth $w(x)$ with $w(l) = 0$ is the solution. We obtain the solution as follows:

$$\boxed{\begin{array}{l}\text{Find } u(x) \text{ among the smooth functions that satisfy } u(l) = \overline{u} \text{ such that} \\ \int_0^l \frac{dw}{dx} AE \frac{du}{dx} dx = (wA\bar{t})_{x=0} + \int_0^l wbdx \qquad \forall w \quad with \quad w(l) = 0\end{array}} \qquad (20)$$

Equation (20) is called the **weak form**. The name states that solutions to the weak form do not need to be as smooth as solutions of the strong form. $\rightarrow$ weaker continuity requirements

You have to keep in mind that the solution satisfying equation (20) is also the solution of the strong counterpart of this equation. Also remember that the trial solutions $u(x)$ must satisfy the **displacement** boudary conditions. This is an **essential** property of the trial solutions and that is why we call those boundary conditions **essential boundary conditions**. The **traction** boundary conditions emanate naturally from equation (20) which means that the trial solutions do not need to be constructed to satisfy the **traction** boundary condition. These boundary conditions are therefore called **natural boundary conditions**.

A trial solution that is **smooth AND** satisfies the **essential boundary conditions** is called **admissible**. A weight function that is **smooth AND** vanishes on **essential boundaries** is **admissible**. When weak forms are used to solve a problem, the trial solutions and weight functions must be **admissible**. Also notice that equation (20) is symmetric in w and u which will lead to a symmetric stiffness matrix. The highest order derivative that appears in this equation is of first order!