3]. Given a random variable Y with characteristic function $\phi(\omega) = E[e^{i\omega Y}]$ and a random process is defined by $X(t) = \cos(\lambda t + Y)$. Show that $\{X(t)\}$ is stationary in the wide sense if $\phi(1) = \phi(2) = 0$

Soln.:

Given $\phi(\omega) = E[e^{i\omega Y}]$ and $X(t) = \cos(\lambda t + Y)$

Since $\phi(1) = 0$

$$\phi(1) = E[e^{iY}] = E[\cos Y + i\sin Y] = 0$$

$$E[\cos Y] + i\,E[\sin Y] = 0 + i0$$

Equating the real & imaginary parts,

$$E[\cos Y] = 0 \quad \text{and} \quad E[\sin Y] = 0 \rightarrow (1)$$

and $\phi(2) = 0$

$$\phi(2) = E[e^{i2Y}] = E[\cos 2Y + i\sin 2Y] = 0$$

$$E[\cos 2Y] + i\,E[\sin 2Y] = 0 + i0$$

Equating the real & imaginary parts,

$$E[\cos 2Y] = 0 \quad \text{and} \quad E[\sin 2Y] = 0 \rightarrow (2)$$

Now, $X(t) = \cos(\lambda t + Y)$

i). $E[X(t)] = E[\cos(\lambda t + Y)]$

$$= E[\cos\lambda t \cos Y - \sin\lambda t \sin Y]$$

$$= E[\cos\lambda t \cos Y] - E[\sin\lambda t \sin Y]$$

$$= \cos\lambda t\, E[\cos Y] - \sin\lambda t\, E[\sin Y]$$

$$= \cos\lambda t\,(0) - \sin\lambda t\,(0) \quad \text{from (1)}$$

$$E[X(t)] = 0$$

ii). $E[X(t)\,X(t+\tau)] = E[\cos(\lambda t + Y)\cos(\lambda(t+\tau) + Y)]$

$$= E[\cos(\lambda t + Y)\cos(\lambda t + \lambda\tau + Y)]$$

$$= \frac{1}{2} E\Big[\cos(\lambda t + Y + \lambda t + \lambda\tau + Y) + \cos(\lambda t + Y - \lambda t - \lambda\tau - Y)\Big]$$

$$= \frac{1}{2} E\left[\cos(2\lambda t + \lambda\tau + 2y) + \cos(-\lambda\tau)\right]$$

$$= \frac{1}{2}\left\{ E\left[\cos(\underbrace{2\lambda t + \lambda\tau}_{A} + \underbrace{2y}_{B})\right] + E\left[\cos(\lambda\tau)\right]\right\}$$

$$= \frac{1}{2}\left\{ E\left[\cos(2\lambda t + \lambda\tau)\cos 2y - \sin(2\lambda t + \lambda\tau)\sin 2y\right] + \cos\lambda\tau\right\}$$

$$= \frac{1}{2}\left\{ E\left[\cos(2\lambda t + \lambda\tau)\cos 2y\right] - E\left[\sin(2\lambda t + \tau\lambda)\sin 2y\right] + \cos\lambda\tau\right\}$$

$$= \frac{1}{2}\left\{ \cos(2\lambda t + \lambda\tau) E[\cos 2y] - \sin(2\lambda t + \lambda\tau) E[\sin 2y] + \cos\lambda\tau\right\}$$

$$= \frac{1}{2}\left\{ 0 + 0 + \cos\lambda\tau\right\}$$

$$= \frac{\cos\lambda\tau}{2}$$ which depends on τ.

$$= R_{XX}(\tau)$$

$\therefore$ $X(t)$ is WSS.