J. The tpm of a markov chain with three states 0, 1, 2 is

$$P = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

and the initial state distribution of the chain is $P(X_0 = i) = \frac{1}{3}, i = 0, 1, 2.$

Find i). $P(X_2 = 2)$

ii) $P(X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2)$

iii). $P(X_2 = 1 / X_0 = 0)$

Soln.

Let $P(X_0 = 0) = \frac{1}{3}$; $P(X_0 = 1) = \frac{1}{3}$; $P(X_0 = 2) = \frac{1}{3}$

i). $P(X_2 = 2)$

Now, $P(X_n = j) = \sum_{i=0}^{j} P(X_n = j / X_0 = i) \cdot P(X_0 = i)$

$$P(X_2 = 2] = \sum_{i=0}^{2} P(X_2 = 2 / X_0 = i)\ P(X_0 = i)$$

$$= P(X_2 = 2 / X_0 = 0)\ P(X_0 = 0) + P(X_2 = 2 / X_0 = 1)\ P(X_0 = 1) + P(X_2 = 2 / X_0 = 2)\ P(X_0 = 2)$$

$$= P_{02}^{(2-0)} P_0^{(0)} + P_{12}^{(2-0)} P_1^{(0)} + P_{22}^{(2-0)} P_2^{(0)}$$

$$= P_{02}^{(2)} P_0^{(0)} + P_{12}^{(2)} P_1^{(0)} + P_{22}^{(2)} P_2^{(0)} \rightarrow (1)$$

Given $P = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix}$

$$P^2 = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix} \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix}$$

$$= \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 0.625 & 0.3125 & 0.0625 \\ 0.3125 & 0.5 & 0.1875 \\ 0.1875 & 0.5625 & 0.25 \end{pmatrix} \end{array}$$

(i) $\Rightarrow$ $P(X_2 = 2) = 0.0625\left(\frac{1}{3}\right) + 0.1875\left(\frac{1}{3}\right) + 0.25\left(\frac{1}{3}\right)$

$= 0.1639$

ii). $P(X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2)$

$= P(X_3 = 1 / X_2 = 2, X_1 = 1, X_0 = 2)\, P(X_2 = 2, X_1 = 1, X_0 = 2$

$= P(X_3 = 1 / X_2 = 2)\, P(X_2 = 2 / X_1 = 1, X_0 = 2)\, P(X_1 = 1, X_0 = 2)$

$= P(X_3 = 1 / X_2 = 2)\, P(X_2 = 2 / X_1 = 1)\, P(X_1 = 1 / X_0 = 2)\, P(X_0 = 2)$

$= P_{21}^{(1)} P_{12}^{(1)} P_{21}^{(1)} P_2^{(0)}$

$= \left(\frac{3}{4}\right)\left(\frac{1}{3}\right)\left(\frac{3}{4}\right)\left(\frac{1}{3}\right)$

$= 0.046$

iii). $P(X_2 = 1 / X_0 = 0) = P_{01}^{(2)} = 0.31$