DEPARTMENT OF AEROSPACE ENGINEERING

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UNIT IV – FINITE VOLUME TECHNIQUES

Multi-stage Time Stepping and its Accuracy

Accuracy

The stencils obtained by the finite volume cell-centered formulation are very similar to the stencils obtained by the analogous finite difference methods. This means that if the grid is sufficiently smooth, such that the cell centers are themselves on a sufficiently smooth grid, i.e. a grid that can be obtained by a continuous mapping from a square grid, the methods discussed in the previous sections are second-order accurate in space in a finite difference sense. This can easily be seen by comparison of the result in Fig. 4.8 with the result obtained by second-order finite difference methods. Since, however, the representation of the solution is done in a piecewise constant way, on an irregular grid the accuracy is formally of first order. In practice, the order is between one and two.

Cell Vertex Formulation

In the cell-vertex formulation, the variables are stored at the vertices of the grid. The control volumes either coincide with cells (non-overlapping case) or consist of a group of cells around a node (overlapping case). Figure 4.5 shows some of the possibilities. In all cases, a linear interpolation of the fluxes is now possible. Therefore, cell-vertex formulations can be second-order accurate in space, irrespective of the irregularity of the grid.

Multi-Stage Time Stepping – Overlapping Control Volumes

For the overlapping cases, the methods discussed in the previous sections can be adapted directly. Very popular nowadays is the formulation of the multi-stage time stepping scheme.

For the overlapping control volumes of Fig. 4.5, the semi-discretization is very similar to (4.17), now involving, however, six or eight surrounding nodes. At solid boundaries, half volumes are formed. The impermeability

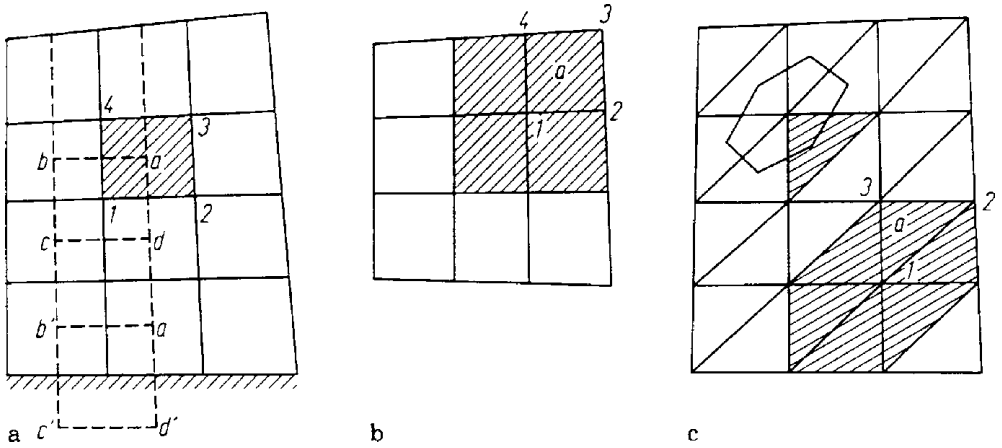

Fig. 4.5 Cell-vertex formulation.

(a): quadrilateral cells, non-overlapping volumes (with interweaving grid); (b): quadrilateral cells, overlapping volumes; (c): triangular cells, overlapping and non-overlapping volumes can be expressed by setting the convective fluxes to zero. Another approach is to treat the control volume as permeable and to impose tangency. This means that, between steps, the normal component of velocity is set equal to zero. Again, to stabilize the scheme, some form of artificial viscosity is necessary. The artificial viscosity is also necessary to eliminate the spurious modes in the solution. Figure 4.12 shows the spurious modes that are possible for the quadrilateral and triangular grids. As in the basic method of Jameson, a blend of second-order smoothing and fourth-order smoothing can be used. Often, the dissipative operator of the cell-centered method is used. This operator is then a sum of terms of form (4.23) for a quadrilateral grid. The method loses then its pure cell-vertex character. The resulting flux balance of inviscid and dissipative terms is then a balance over a control volume centered around a vertex as shown in Fig. 4.13. Such a control volume is called a dual control volume

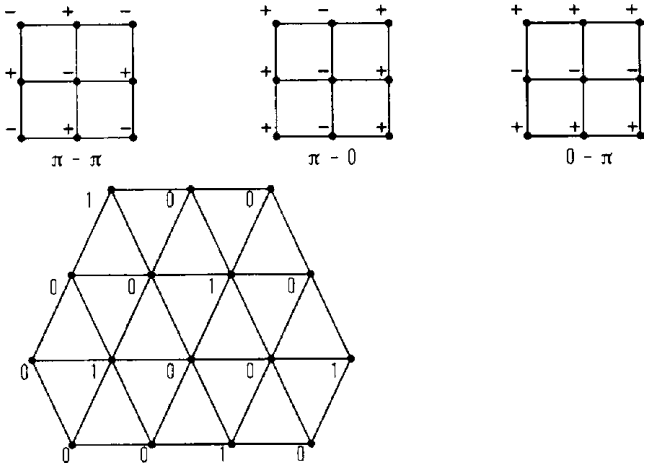

Fig. 4.12 Spurious modes for cell-vertex central discretization

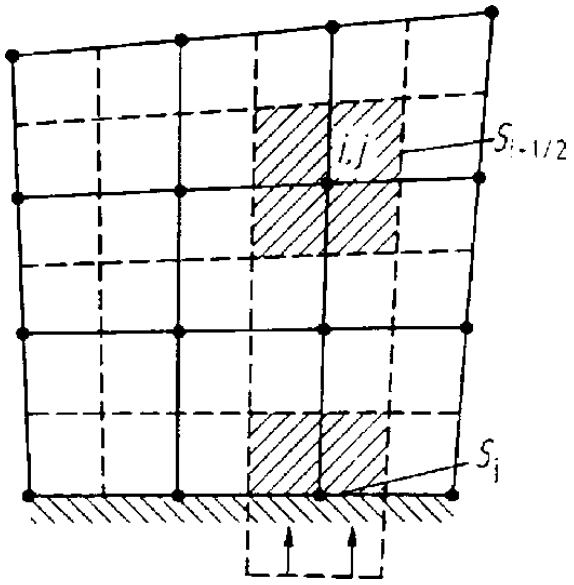

Fig. 4.13 Vertex-based FVM

The inviscid flux balance over the dual control volume can be defined as one-fourth of the flux balance over the volume formed by the four surrounding cells. Strictly, the method then becomes a vertex-centered or vertex-based method according to the terminology introduced in Sect. 4.3. A pure cell-vertex method can be obtained by changing the construction of the dissipator. The same methodology as for the cell-centered method is used, but summations now run over cells surrounding a node rather than over surrounding nodes.

This means that differences in values used in the expression (4.27) have to be modified.

For instance $U_j - U_i$ is to be replaced by

$$\tfrac{1}{2}(U_{j1} + U_{j2}) - U_i$$

or

$$\tfrac{1}{3}(U_{j1} + U_{j2} + U_{j3}) - U_i \quad \text{or} \quad \tfrac{1}{2}(U_{j1} + U_{j3}) - U_i$$

For triangular and quadrilateral cells respectively, where j1, j2, and j3 denote the nodes not coinciding with node i of the surrounding cells. Also the scaling factors

$\zeta_{i,j}$ and the weight factors $\varepsilon(2)$

$_{i,j}$, $\varepsilon(4)$

$_{i,j}$ now involves maxima over all nodes of a cell.

The foregoing smoothing procedure is conservative in the sense that the content of a cell is not changed by the dissipator. The formula for the update of a node is the sum of the contributions of the surrounding cells. The update coming from the inviscid flux balance over a cell is

modified by the dissipator. The modification is such that the flux balance over a cell can be seen as unequally distributed to its vertices but with a sum of distribution factors equal to one. So, the dissipator acts as a redistributor of the flux balances of the cells.

The pure cell-vertex method is not very often used. Most researchers employ the first described vertex-based approach but call it a cell-vertex method. The pure cell-vertex method has an obvious difficulty on a triangular grid. Since there are about twice as many cells as nodes, it is not possible to satisfy the flux balances of individual cells and reach a steady state. Even on a structured grid, it is rather delicate to satisfy flux balances over individual cells.

Lax-Wendroff Time-Stepping Non-Overlapping Control Volumes

For the non-overlapping case, a Lax-Wendroff variant exists due to Ni, developed in 1981. It requires the use of a second set of control volumes centered around the nodes,

obtained in the way shown in Fig. 4.5a. Ni‘s variant starts from the Lax-Wendroff formulation (4.8), (4.9). Without loss of accuracy in (4.9), $\partial f/\partial t$ can be replaced by a first-order accurate difference $\Delta f/\Delta t$. The result is

$$u^{n+1} \approx u^n - \Delta t \frac{\partial f^n}{\partial x} - \frac{\Delta t}{2} \frac{\partial}{\partial x}(\Delta f)$$

In two dimensions, on the Euler equations, this is

$$\Omega_{i,j}(U^{n+1} - U^n) = -\Delta t \left[\int (f^n\, dy - g^n\, dx) + \tfrac{1}{2} \int (\Delta f\, dy - \Delta g\, dx) \right]$$

On the quadrilateral grid of Fig. 5a, the method is then as follows. Based on the cell 1-2-3-4, using an Euler step, i.e. a step forward in time, a first-order approximation of the increment of the flux vectors is obtained from

$$\Omega_a \Delta U_a = -\Delta t \int_{1234} (f\, dy - g\, dx)^n$$

And

$$\Delta f_a = A\, \Delta U_a, \qquad \Delta g_a = B\, \Delta U_a$$

Where A and B are the Jacobians of the flux vectors f and g to U. A and B are taken to be the mean values of the Jacobians evaluated at the nodes 1, 2, 3, and 4. The area-weighted mean value of the first-order increments given by (4.29) over the four cells surrounding node 1, gives a first-order increment for the dependent variables:

ΔU_1^1

The discretization of (4.28) on the cell is then:

$$\Omega_1(U_1^{n+1} - U_1^n) = \Omega_1 \, \Delta U_1^1 - \tfrac{1}{2} \Delta t \int_{abcd} (\Delta f \, dy - \Delta g \, dx)$$

The spatial integration is again taken to be piecewise linear. The CFL restriction for the time step, given by Ni is

$$\Delta t \leq \min\left(\frac{\Delta x}{|u| + c}, \quad \frac{\Delta y}{|u| + c} \right)$$

with

$$\Delta x = \frac{x_{i+1,j} - x_{i-1,j}}{2} \quad , \qquad \Delta y = \frac{y_{i,j+1} - y_{i,j-1}}{2}$$

The boundary conditions at solid boundaries for the first step (4.29) can be implemented by setting convective fluxes equal to zero, as in the previous methods. In the second step (4.30), a half-volume is needed around a boundary node. This half volume can be seen to be half the complete volume shown in Fig. 4.5a. Step (4.30) can be done by setting the first-order changes in the fictitious cells equal to zero. So, the boundary node only receives both first-order and second-order contributions from the inward cells and b. As a consequence, for a boundary node, there is no implicit imposition of impermeability in step (4.30). Tangency is then imposed afterward by setting the normal component of the velocity equal to zero.

It is to be remarked that, although an intermediate grid is used, the Ni-method is a true cell-vertex method. Indeed, if the flux balance of a cell is satisfied, there is no contribution to both first- and second-order terms, and flow parameters are not changed. Therefore step (4.30) often is called the distribution step since its function can be seen to be the distribution of changes in the control volumes to the nodes.

As already mentioned, in a triangular grid, there are about twice as many cells as nodes. This means that in a cell-vertex formulation, flux balances cannot be satisfied for all cells. The steady-state result of a cell-vertex time stepping scheme then corresponds to some combinations of flux balances being zero. In a quadrilateral grid, all flux balances can be satisfied at a steady state. We also note that the distribution of the changes in the control volumes for triangular cells can be done with upwind methods.