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UNIT IV – FINITE VOLUME TECHNIQUES

Central and Upwind-type Discretization Techniques

Central Type Discretizations

The adaptation of a Lax-Wendroff time-stepping or a multi-stage time-stepping, as discussed for the cell-centered FVM, to the vertex-based FVM is straightforward. The formulations obtained with both methods are very similar, except at solid boundaries.

Upwind Type Discretizations

As an example of an upwind discretization, we treat here the flux-difference splitting technique introduced by Roe.

The flux through a surface (i + 1/2) of the control volume on Fig. 4.13 can be written as

$$F_{i+1/2} = \Delta y_{i+1/2} f_{i+1/2} - \Delta x_{i+1/2} g_{i+1/2}$$

where $f_{i+1/2}$ and $g_{i+1/2}$ have to be defined using the values of the flux vectors in the nodes (i,j) and (i+1, j). We switch here to the classic finite difference notation using halves in the subscripts to denote intermediate points. Also, non-varying subscripts are not written. We denote by Fi the value of $F_{i+1/2}$ using the function values in (i,j) and by F_{i+1}, the value using the function values in (i+1, j). The flux (4.31) can be written as

$$F_{i+1/2} = \Delta s_{i+1/2}(n_x f_{i+1/2} + n_y g_{i+1/2})$$

with

$$n_x = \Delta y_{i+1/2}/\Delta s_{i+1/2}, \qquad n_y = -\Delta x_{i+1/2}/\Delta s_{i+1/2}, \qquad \Delta s^2_{i+1/2} = \Delta x^2_{i+1/2} + \Delta y^2_{i+1/2}$$

To define an upwind flux, we consider the flux-difference

$$\Delta F_{i,i+1} = \Delta s_{i+1/2}(n_x \Delta f_{i,i+1} + n_y \Delta g_{i,i+1})$$

where

$$\Delta f_{i,i+1} = f_{i+1,j} - f_{i,j}, \qquad \Delta g_{i,i+1} = g_{i+1,j} - g_{i,j}$$

For the construction of the flux, it is essential that the linear combination of Δf and Δg in (4.33) can be written as

$$\Delta\phi = n_x \Delta f + n_y \Delta g = A \Delta U$$

where A is a discrete Jacobian matrix with similar properties as the analytical Jacobians of the flux vectors. This means that the eigenvalues of A are real and that the matrix has a complete set of eigenvectors. Of course, for consistency, the eigenvalues and eigenvectors should be approximations of the eigenvalues and eigenvectors of the linear combination of the analytical Jacobians. The construction of the discrete Jacobian is not unique and many formulations have been proposed after the first formulation by Roe. For the numerical illustration later in this section, we use the formulation by the author. The algebraic manipulations in the construction of the discrete Jacobian are not relevant for a principal discussion of the methodology and we do not describe these here.

The matrix A can be split into positive and negative parts by

$$A^+ = R\Lambda^+ L, \qquad A^- = R\Lambda^- L$$

Where R and L denote the right and left eigenvector matrices in orthonormal form and where

$$\Lambda^+ = \mathrm{diag}(\lambda_1^+, \lambda_2^+, \lambda_3^+, \lambda_4^+), \qquad \Lambda^- = \mathrm{diag}(\lambda_1^-, \lambda_2^-, \lambda_3^-, \lambda_4^-)$$

$$\text{with } \lambda_i^+ = \max(\lambda_i, 0),\ \lambda_i^- = \min(\lambda_i, 0).$$

Positive and negative matrices denote matrices with, respectively, non-negative and non-positive eigenvalues.

This allows a splitting of the flux-difference (4.34) by

$$\Delta\phi = A^+ \Delta U + A^- \Delta U$$

As a consequence (4.33) can be written as

$$\Delta F_{i,i+1} = F_{i+1} - F_i = \Delta s_{i+1/2} A_{i,i+1} \Delta U_{i,i+1}$$

Where the matrix $A_{i,i+1}$ can be split into positive and negative parts. The absolute value of the flux-difference is defined by

$$|\Delta F_{i,i+1}| = \Delta s_{i+1/2} (A^+_{i,i+1} - A^-_{i,i+1}) \Delta U_{i,i+1}$$

Based on (4.36) an upwind definition of the flux is

$$F_{i+1/2} = 1/2 \left[F_i + F_{i+1} - |\Delta F_{i,i+1}| \right]$$

That this represents an upwind flux can be verified by writing (4.37) in either of the two following ways, which are completely equivalent:

$$F_{i+1/2} = F_i + 1/2 \Delta F_{i,i+1} - 1/2 |\Delta F_{i,i+1}| = F_i + \Delta s_{i+1/2} A^-_{i,i+1} \Delta U_{i,i+1}$$

$$F_{i+1/2} = F_{i+1} - 1/2 \Delta F_{i,i+1} - 1/2 |\Delta F_{i,i+1}| = F_{i+1} - \Delta s_{i+1/2} A^+_{i,i+1} \Delta U_{i,i+1}$$

Indeed, when $A_{i,i+1}$ has only positive eigenvalues, the flux $F_{i+1/2}$ is taken to be Fi, and when $A_{i,i+1}$ has only negative eigenvalues, the flux $F_{i+1/2}$ is taken to be F_{i+1}.

The fluxes on the other surfaces of the control volume $S_{i-1/2}$, $S_{j+1/2}$, and $S_{j-1/2}$ can be treated similarly to the flux on the surface $S_{i+1/2}$. With (4.38) and (4.39), the flux balance on the control volume of Fig. 4.13 can be brought into the form.

$$\Delta s_{i+1/2} A^-_{i,i+1} [U_{i+1} - U_i] + \Delta s_{i-1/2} A^+_{i,i-1} [U_i - U_{i-1}] + \Delta s_{j+1/2} A^-_{j,j+1} [U_{j+1} - U_j] + \Delta s_{j-1/2} A^+_{j,j-1} [U_j - U_{j-1}] = 0$$

or

$$CU_{i,j} = \Delta s_{i-1/2} A^+_{i,i-1} U_{i-1,j} + \Delta s_{i+1/2} (-A^-_{i,i+1}) U_{i+1,j} + \Delta s_{j-1/2} A^+_{j,j-1} U_{i,j-1} + \Delta s_{j+1/2} (-A^-_{j,j+1}) U_{i,j+1}$$

Where C is the sum of the matrix coefficients on the right-hand side. The matrix coefficients in (4.41) have non-negative eigenvalues. The positivity of the coefficients on the right-hand side of (4.41) and the (weak) dominance of the central coefficient guarantee that the solution can be obtained by a collective variant of any scalar relaxation method. A collective variant means that in each node all components of the vector of dependent variables U are relaxed simultaneously.

In order to illustrate the boundary treatment, we consider now the half-volume on a solid boundary as shown in Fig. 4.13. This half-volume can be seen as the limit of a complete volume in which one of the sides tends to the boundary.

The flux on the side Sj of the control volume at the solid boundary can be expressed By

$$F_j - \Delta s_j A^+_{i,j}(U_j - U_{j-1})$$

where the matrix Ai,j is calculated with the function values in the node (i,j). With the definition (4.42), the flux balance on the control volume takes the form (4.40) in which a node outside the domain comes in. This node, however, can be eliminated.

It is easily seen that on a solid boundary, three combinations of (4.42) exist, eliminating the outside node. The combinations are the left eigenvectors corresponding to the zero eigenvalues in A+i,j. These equations are to be supplemented by the boundary condition of tangency.

As an illustration, Fig. 4.14 shows the solution obtained by the previous method for the test case of Fig. 4.7 under the same conditions as for Fig. 4.8. Comparison of the upwind result with the central result shows the superiority of the upwind calculation to the sharpness of the shock.

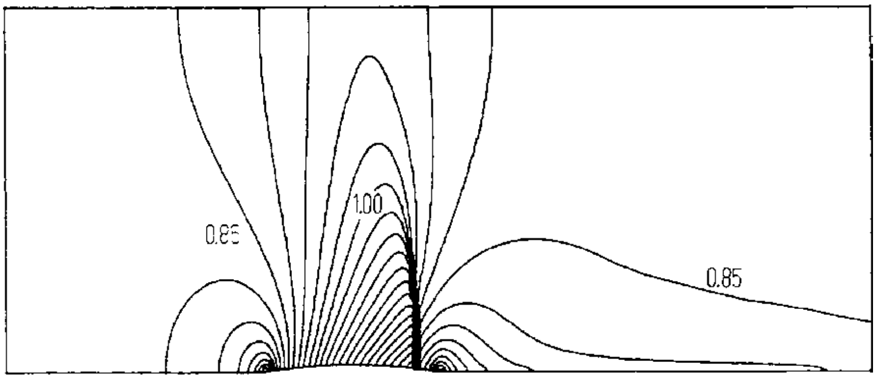

Fig. 4.14 IsoMachlines obtained by a vertex-based upwind FVM

In the above, the upwind discretization is used in first-order form. For more complex flows, of course, at least second-order accuracy is needed. In this introductory text, we prefer not to enter the discussion of higher-order upwinding.