Solution of Linear System

There are two types of methods to solve linear algebraic equations

(i) Direct Method:

(a) Gauss Elimination Method

(b) Gauss Jordan Method

(ii) Indirect Method (or) Iterative Method:

(a) Gauss Jacobi Method

(b) Gauss Seidel Method

Gauss Elimination Method:

Let us consider the 'n' linear equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

where a_{ij} and b_i are known constants and x_i's are unknowns

The above eqn. is equivalent to $AX = B$

where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

Now our aim is to reduce the augmented matrix $[A, B]$ to upper triangular matrix.

Augmented matrix is

$$[A,B] = \begin{pmatrix} a_{11} & a_{12} & \cdot & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdot & a_{2n} & b_2 \\ \vdots & & & & \\ a_{n1} & a_{n2} & \cdot & a_{nn} & b_n \end{pmatrix}$$

which is reduced to upper triangular matrix, as,

$$\begin{pmatrix} a_{11} & a_{12} & \cdot & a_{1n} & b_1 \\ 0 & b_{22} & \cdot & b_{2n} & c_2 \\ \vdots & \vdots & & & \\ 0 & 0 & & & d_n \end{pmatrix}$$

By back substitution method we get the values for $x_n, x_{n-1}, \ldots, x_2, x_1$.

① Solve the system of equations by Gaussian elimination method.

$$10x - 2y + 3z = 23$$
$$2x + 10y - 5z = -33$$
$$3x - 4y + 10z = 41$$

The given system is equivalent to $AX = B$

(ii) $$\begin{pmatrix} 10 & -2 & 3 \\ 2 & 10 & -5 \\ 3 & -4 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 23 \\ -33 \\ 41 \end{pmatrix}$$

Now $$[A, B] = \begin{bmatrix} 10 & -2 & 3 & 23 \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{bmatrix}$$

Let us reduce augmented matrix $[A, B]$ to upper triangular matrix.

Step 1: Fix the first row, change 2 & 3 row with row 1

$$[A, B] \sim \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 10.4 & -5.6 & -37.6 \\ 0 & -3.4 & 9.1 & 34.1 \end{bmatrix} \begin{array}{l} \\ R_2 \leftrightarrow R_2 - \frac{2}{10} R_1 \\ R_3 \leftrightarrow R_3 - \frac{3}{10} R_1 \end{array}$$

Step 2: Fix 1 & 2 row, change 3 row with 2nd row

$$\sim \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 10.4 & -5.6 & -37.6 \\ 0 & 0 & 7.26 & 21.80 \end{bmatrix} \begin{array}{l} \\ \\ R_3 \leftrightarrow R_3 - \left(\frac{-3.4}{10.4}\right) R_2 \end{array}$$

which is an upper triangular matrix.

Step 3: Back substitution.

We get, $7.26z = 21.80 \Rightarrow z = 3$

$10.4y - 5.6z = -37.6 \Rightarrow y = -2$

$10x - 2y + 3z = 23 \Rightarrow x = 1$

checking: $10x - 2y$
$10(1) - 2(-2)$

Hence soln. is $x = 1,\ y = -2,\ z = 3$

2) Solve the system of equations by Gauss-elimination method.

$$5x_1 + x_2 + x_3 + x_4 = 4$$
$$x_1 + 7x_2 + x_3 + x_4 = 12$$
$$x_1 + x_2 + 6x_3 + x_4 = -5$$
$$x_1 + x_2 + x_3 + 4x_4 = -6$$

The gn. system is equivalent to $AX = B$

(i.e.) $$\begin{bmatrix} 5 & 1 & 1 & 1 \\ 1 & 7 & 1 & 1 \\ 1 & 1 & 6 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 12 \\ -5 \\ -6 \end{pmatrix}$$

Now $[A, B] = \begin{bmatrix} 5 & 1 & 1 & 1 & 4 \\ 1 & 7 & 1 & 1 & 12 \\ 1 & 1 & 6 & 1 & -5 \\ 1 & 1 & 1 & 4 & -6 \end{bmatrix}$

Let us reduce augmented matrix to upper triangular matrix.

Step 1: Fix row 1, change 2, 3, 4 row with row 1

$$[A,B] \sim \begin{bmatrix} 5 & 1 & 1 & 1 & 4 \\ 0 & 6.8 & 0.8 & 0.8 & 11.2 \\ 0 & 0.8 & 5.8 & 0.8 & -5.8 \\ 0 & 0.8 & 0.8 & 3.8 & -6.8 \end{bmatrix} \begin{matrix} \\ R_2 \leftrightarrow R_2 - \frac{1}{5}R_1 \\ R_3 \leftrightarrow R_3 - \frac{1}{5}R_1 \\ R_4 \leftrightarrow R_4 - \frac{1}{5}R_1 \end{matrix}$$

Step 2: Fix row 1 & 2, change 3 & 4 row with row 2.

$$\sim \begin{bmatrix} 5 & 1 & 1 & 1 & 4 \\ 0 & 6.8 & 0.8 & 0.8 & 11.2 \\ 0 & 0 & 5.70 & 0.70 & -7.11 \\ 0 & 0 & \underset{0.70}{(-5)} & \underset{3.70}{(3)} & \underset{-8.11}{(-7)} \end{bmatrix} \begin{matrix} \\ \\ R_3 \leftrightarrow R_3 - \frac{0.8}{6.8}R_2 \\ R_4 \leftrightarrow R_4 - \frac{0.8}{6.8}R_2 \end{matrix}$$

Step 3: Fix row 1, 2 & 3, Change 4th row with row 3

$$\sim \begin{bmatrix} 5 & 1 & 1 & 1 & 4 \\ 0 & 6.8 & 0.8 & 0.8 & 11.2 \\ 0 & 0 & 5.70 & 0.70 & -7.11 \\ 0 & 0 & 0 & 3.61 & -7.23 \end{bmatrix} R_4 \leftrightarrow R_4 - \left(-\frac{5}{5.70}\right)R_3$$

Step 4: Back Substitution.

we get $3.61\, x_4 = -7.23 \Rightarrow x_4 = -2.00$

$5.70\, x_3 + 0.70\, x_4 = -7.11 \Rightarrow x_3 = -1.00$

$6.8\, x_2 + 0.8\, x_3 + 0.8\, x_4 = 11.2 \Rightarrow x_2 = 2$

$5x_1 + x_2 + x_3 + x_4 = 4 \Rightarrow x_1 = 1$

$\therefore$ Soln. is $x_1 = 1,\ x_2 = 2,\ x_3 = -1,\ x_4 = -2$.