Taylor's Series:

If $f(z)$ is analytic inside a circle C with centre at $z=a$, then $f(z)$ can be expressed as,

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots + \frac{(z-a)^n}{n!} f^n(a) + \cdots$$

which is convergent at every point inside C. This is called Taylor's series of $f(z)$ about $z=a$.

Note:

The Taylor's series of $f(z)$ about the point $z=0$ is given by,

$$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \cdots + \frac{z^n}{n!} f^n(0) + \cdots$$

1) Expand $f(z) = \log(1+z)$ as Taylor's series about $z=0$ if $|z|<1$.

Soln:

$f(z) = \log(1+z)$ $\qquad$ $f(0) = \log 1 = 0$

$f'(z) = \frac{1}{1+z}$ $\qquad$ $f'(0) = \frac{1}{1} = 1$

$f''(z) = \frac{-1}{(1+z)^2}$ $\qquad$ $f''(0) = \frac{-1}{1} = -1$

$f'''(z) = \frac{2}{(1+z)^3}$ $\qquad$ $f'''(0) = \frac{2}{1} = 2$

Taylor's series about $z=0$ is given by

$$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0) + \dots$$

$$= 0 + \frac{z}{1!}(1) + \frac{z^2}{2!}(-1) + \frac{z^3}{3!}(2) + \dots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \dots$$

(2) Expand $f(z) = \cos z$ about $z = \pi/3$ in Taylor's series.

Soln:

$f(z) = \cos z \qquad f(\pi/3) = \cos \pi/3 = 1/2$

$f'(z) = -\sin z \qquad f'(\pi/3) = -\sin \pi/3 = -\sqrt{3}/2$

$f''(z) = -\cos z \qquad f''(\pi/3) = -\cos \pi/3 = -1/2$

$f'''(z) = \sin z \qquad f'''(\pi/3) = \sin \pi/3 = \sqrt{3}/2$

Taylor's series is,

$$f(z) = f\left(\frac{\pi}{3}\right) + \frac{(z-\pi/3)}{1!} f'\left(\frac{\pi}{3}\right) + \frac{(z-\pi/3)^2}{2!} f''\left(\frac{\pi}{3}\right) + \dots$$

$$= \frac{1}{2} + \frac{(z-\pi/3)}{1!}\left(-\frac{\sqrt{3}}{2}\right) + \frac{(z-\pi/3)^2}{2!}\left(-\frac{1}{2}\right) + \frac{(z-\frac{\pi}{3})^3}{3!}\left(\frac{\sqrt{3}}{2}\right) + \dots$$

$$= \frac{1}{2} - \frac{(z-\pi/3)}{1!}\left(\frac{\sqrt{3}}{2}\right) - \frac{(z-\pi/3)^2}{2!}\left(\frac{1}{2}\right) + \frac{(z-\frac{\pi}{3})^2}{3!}\left(\frac{\sqrt{3}}{2}\right) + \dots$$

(3) Expand $f(z) = \sin z$ in a Taylor's Series about $z = \pi/4$.

Soln:

$$f(z) = \sin z \qquad f(\pi/4) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$f'(z) = \cos z \qquad f'(\pi/4) = \cos \pi/4 = 1/\sqrt{2}$$

$$f''(z) = -\sin z \qquad f''(\pi/4) = -\sin \pi/4 = -1/\sqrt{2}$$

$$f'''(z) = -\cos z \qquad f'''(\pi/4) = -\cos \pi/4 = -1/\sqrt{2}$$

Taylor's Series formula for $z=a$

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a) + \cdots$$

At $z = \pi/4$

$$f(z) = f\left(\frac{\pi}{4}\right) + \frac{(z-\pi/4)}{1!} f'\left(\frac{\pi}{4}\right) + \frac{(z-\pi/4)^2}{2!} f''\left(\frac{\pi}{4}\right) + \frac{\left(z-\frac{\pi}{4}\right)^3}{3!} f'''\left(\frac{\pi}{4}\right) + \cdots$$

$$f(z) = \frac{1}{\sqrt{2}} + \left(z-\frac{\pi}{4}\right)\frac{1}{\sqrt{2}} + \frac{\left(z-\frac{\pi}{4}\right)^2}{2} \cdot \left(\frac{-1}{\sqrt{2}}\right) + \frac{\left(z-\frac{\pi}{4}\right)^3}{6} \cdot \left(\frac{-1}{\sqrt{2}}\right) + \cdots$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(z-\frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(z-\frac{\pi}{4}\right)^2 - \frac{1}{6\sqrt{2}}\left(z-\frac{\pi}{4}\right)^3 + \cdots$$