Cauchy's Residue theorem:

Statement only

[If $f(z)$ is analytic at all points inside and on a simple closed curve c except at a finite number of points $z_1, z_2, z_3, \ldots z_n$ inside c, then

$$\int_c f(z)\,dz = 2\pi i \left[\text{Sum of residues of } f(z) \text{ at } z_1, z_2, \ldots z_n \right]$$]

proof:

Given that $f(z)$ is not analytic only at $z_1, z_2, \ldots z_n$.

Draw the non-intersecting small circles $c_1, c_2, \ldots c_n$ with centres at $z_1, z_2, \ldots z_n$ & radii $\rho_1, \rho_2, \ldots \rho_n$ lying wholly inside C.

Then $f(z)$ is analytic in the region between C and $C_1, C_2, \dots C_n$.

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \dots + \int_{C_n} f(z)\,dz \rightarrow (1)$$

Now $z_1, z_2, \dots z_n$ are the singular points of $f(z)$.

$\therefore \{\text{Res } f(z)\}_{z=z_i}$ = the coef of $\frac{1}{z-z_i}$ in the Laurent's series of $f(z)$ about $z = z_i$ (by defn of residues)

$$= b_1 = \frac{1}{2\pi i}\int_{C_i} \frac{f(z)}{(z-z_i)^{1-1}}\,dz$$

$$\left[\text{Since } b_n = \frac{1}{2\pi i}\int_{C_i} \frac{f(z)}{(z-z_i)^{1-n}}\,dz\right]$$

$$= \frac{1}{2\pi i}\int_{C_i} \frac{f(z)}{z-z_i}\,dz.$$

$$= \frac{1}{2\pi i}\int_{C_i} f(z)\,dz$$

$$\Rightarrow \int_{C_i} f(z)\,dz = 2\pi i\,\{\text{Res } f(z)\}_{z=z_i} \rightarrow (2)$$

From (1) & (2),

$$\int_C f(z)\,dz = 2\pi i\{\text{Res } f(z)\}_{z=z_1} + 2\pi i\{\text{Res } f(z)\}_{z=z_2} + \dots + 2\pi i\{\text{Res } f(z)\}_{z=z_n}$$

$$= 2\pi i\left\{[\text{Res } f(z)]_{z=z_1} + [\text{Res } f(z)]_{z=z_2} + \dots + [\text{Res } f(z)]_{z=z_n}\right\}$$

$= 2\pi i \{$ Sum of residues of $f(z)$ at $z = z_1, z_2, \dots z_n \}$

$= 2\pi i \sum_i R_i$

(1) Evaluate $\int_C \frac{e^z}{(z+2)(z+1)^2} dz$ where c is the circle $|z| = 3$.

Soln:

Let $f(z) = \frac{e^z}{(z+2)(z+1)^2}$

The poles of $f(z)$ are given by,

$(z+2)(z+1)^2 = 0$

$\Rightarrow z = -1$ is a pole of order 2.

$\Rightarrow z = -2$ is a pole of order 1.

Given : $|z| = 3$

$z = -1 \Rightarrow |z| = 1 < 3$, lies inside c

$z = -2 \Rightarrow |z| = 2 < 3$, lies inside c

$$\{\text{Res } f(z)\}_{z=-2} = \underset{z \to -2}{Lt} \cancel{(z+2)} \cdot \frac{e^2}{\cancel{(z+2)}(z+1)^2}$$

$$= \frac{e^2}{(-2+1)^2}$$

$$= e^2$$

$$\{\text{Res } f(z)\}_{z=-1} = \underset{z \to -1}{Lt} \frac{1}{1!} \frac{d}{dz} \left[(z+1)^2 \frac{e^2}{(z+1)^2 (z+2)} \right]$$

$$= \underset{z \to -1}{Lt} \frac{d}{dz} \cdot \frac{e^2}{(z+2)}$$

$$= \underset{z \to -1}{Lt} \frac{-e^2}{(z+2)^2} = \frac{-e^2}{(-1+2)^2} = -e^2$$

$\therefore$ By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \,[\text{Sum of the residues of } f(z) \text{ at the poles which lie inside } C]$$

$$\int_C \frac{e^2\,dz}{(z+2)(z+1)^2} = 2\pi i\,(e^2 - e^2) = 0.$$

(2) Evaluate $\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$ where C is the circle $|z| = 3/2$.

Soln:

Let $f(z) = \frac{4-3z}{z(z-1)(z-2)}$

The poles of $f(z)$ are,

$z(z-1)(z-2) = 0$

$z=0,\ z=1,\ z=2$ are poles of order 1.

$z = 0 \Rightarrow |z| = 0 < 3/2$ lies inside C

$z = 1 \Rightarrow |z| = 1 < 3/2$ " "

$z = 2 \Rightarrow |z| = 2 > 3/2$ lies outside C.

$$\{\text{Res } f(z)\}_{z=0} = \lim_{z\to 0} (z-0)\frac{4-3z}{z(z-1)(z-2)}$$

$$= \frac{4}{(-1)(-2)} = 2$$

$$\{\text{Res } f(z)\}_{z=1} = \lim_{z\to 1} (z-1)\frac{4-3z}{z(z-1)(z-2)}$$

$$= \frac{1}{-1} = -1$$

$\{\text{Res } f(z)\}_{z=2} = 0$ (lies outside c)

By Cauchy's residue theorem,

$\int_C f(z)\,dz = 2\pi i$ [Sum of residues of $f(z)$ at the poles which lie inside c]

$= 2\pi i\,(2-1)$

$= 2\pi i$