



DEPARTMENT OF MATHEMATICS

Cauchy's Residue theorem:

[If $f(z)$ is analytic at all points inside and on a simple closed curve c except at a finite number of points $z_1, z_2, z_3, \dots, z_n$ inside c , then

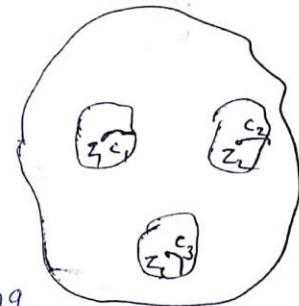
Statement only

$$\int_c f(z) dz = 2\pi i \left[\text{Sum of residues of } f(z) \text{ at } z_1, z_2, \dots, z_n \right]$$

Proof:

Given that $f(z)$ is not analytic only at z_1, z_2, \dots, z_n .

Draw the non-intersecting small circles C_1, C_2, \dots, C_n with centres at z_1, z_2, \dots, z_n & radii $\rho_1, \rho_2, \dots, \rho_n$ lying wholly inside c .





SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Approved by AICTE, New Delhi, Affiliated to Anna University, Chennai

Accredited by NAAC-UGC with 'A++' Grade (Cycle III) &

Accredited by NBA (B.E - CSE, EEE, ECE, Mech & B.Tech.IT)

COIMBATORE-641 035, TAMIL NADU



DEPARTMENT OF MATHEMATICS

Then $f(z)$ is analytic in the region between C and C_1, C_2, \dots, C_n .

$$\int_C f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz + \dots + \int_{C_n} f(z) dz \rightarrow (1)$$

Now z_1, z_2, \dots, z_n are the singular points of $f(z)$.

$\therefore \{ \text{Res } f(z) \}_{z=z_i}$ = the coef of $\frac{1}{z-z_i}$ in the Laurent's series of $f(z)$ about $z = z_i$ (by defn of residues)

$$= b_1 = \frac{1}{2\pi i} \int_{C_i} \frac{f(z)}{(z-z_i)^{-1}} dz$$
$$\left[\text{Since } b_n = \frac{1}{2\pi i} \int_{C_i} \frac{f(z)}{(z-z_i)^{-n}} dz \right]$$
$$= \frac{1}{2\pi i} \int_{C_i} \frac{f(z)}{z-z_i} dz$$
$$= \frac{1}{2\pi i} \int_{C_i} f(z) dz$$
$$\Rightarrow \int_{C_i} f(z) dz = 2\pi i \{ \text{Res } f(z) \}_{z=z_i} \rightarrow (2)$$

From (1) & (2),

$$\int_C f(z) dz = 2\pi i \{ \text{Res } f(z) \}_{z=z_1} + 2\pi i \{ \text{Res } f(z) \}_{z=z_2} + \dots + 2\pi i \{ \text{Res } f(z) \}_{z=z_n}$$
$$= 2\pi i \left\{ [\text{Res } f(z)]_{z=z_1} + [\text{Res } f(z)]_{z=z_2} + \dots + [\text{Res } f(z)]_{z=z_n} \right\}$$



DEPARTMENT OF MATHEMATICS

$$= 2\pi i \left\{ \text{Sum of residues of } f(z) \text{ at } z = z_1, z_2, \dots, z_n \right\}$$

$$= 2\pi i \sum R_i$$

① Evaluate $\int_c \frac{e^z}{(z+2)(z+1)^2} dz$ where c is the circle

$$|z| = 3.$$

Soln:

$$\text{Let } f(z) = \frac{e^z}{(z+2)(z+1)^2}$$

The poles of $f(z)$ are given by,

$$(z+2)(z+1)^2 = 0$$

$\Rightarrow z = -1$ is a pole of order 2.

$\Rightarrow z = -2$ is a pole of order 1.

Given: $|z| = 3$.

$z = -1 \Rightarrow |z| = 1 < 3$, lies inside c

$z = -2 \Rightarrow |z| = 2 < 3$, lies inside c

$$\left\{ \text{Res } f(z) \right\}_{z=-2} = \lim_{z \rightarrow -2} (z+2) \cdot \frac{e^z}{(z+2)(z+1)^2}$$

$$= \frac{e^z}{(-2+1)^2}$$

$$= \frac{e^{-2}}{(-1)^2} = e^{-2}$$

$$\left\{ \text{Res } f(z) \right\}_{z=-1} = \lim_{z \rightarrow -1} \frac{1}{1!} \frac{d}{dz} \left[\frac{(z+1)^2 e^z}{(z+1)^2 (z+2)} \right]$$

$$= \lim_{z \rightarrow -1} \frac{d}{dz} \frac{e^z}{z+2}$$

$$= \lim_{z \rightarrow -1} \frac{-e^z}{(z+2)^2} = \frac{-e^{-1}}{(-1+2)^2} = -e^{-1}$$



SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Approved by AICTE, New Delhi, Affiliated to Anna University, Chennai

Accredited by NAAC-UGC with 'A++' Grade (Cycle III) &

Accredited by NBA (B.E - CSE, EEE, ECE, Mech & B.Tech.IT)

COIMBATORE-641 035, TAMIL NADU



DEPARTMENT OF MATHEMATICS

∴ By Cauchy's Residue theorem,

$$\int_C f(z) dz = 2\pi i \left[\text{Sum of the residues of } f(z) \text{ at the poles which lie inside } C \right]$$

$$\int_C \frac{e^z dz}{(z+2)(z+1)^2} = 2\pi i (e^2 - e^2) = 0$$

2) Evaluate $\int_C \frac{4-3z}{z(z-1)(z-2)} dz$ where C is the

Circle $|z| = 3/2$.

Soln:

$$\text{Let } f(z) = \frac{4-3z}{z(z-1)(z-2)}$$

The poles of $f(z)$ are,

$$z(z-1)(z-2) = 0$$

$z=0, z=1, z=2$ are poles of order 1.

$$z=0 \Rightarrow |z| = 0 < 3/2 \text{ lies inside } C$$

$$z=1 \Rightarrow |z| = 1 < 3/2 \text{ " " "}$$

$$z=2 \Rightarrow |z| = 2 > 3/2 \text{ lies outside } C.$$

$$\begin{aligned} \left\{ \text{Res } f(z) \right\}_{z=0} &= \lim_{z \rightarrow 0} (z-0) \frac{4-3z}{z(z-1)(z-2)} \\ &= \frac{4}{(-1)(-2)} = 2 \end{aligned}$$

$$\begin{aligned} \left\{ \text{Res } f(z) \right\}_{z=1} &= \lim_{z \rightarrow 1} (z-1) \frac{4-3z}{z(z-1)(z-2)} \\ &= \frac{1}{-1} = -1 \end{aligned}$$



SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Approved by AICTE, New Delhi, Affiliated to Anna University, Chennai

Accredited by NAAC-UGC with 'A++' Grade (Cycle III) &

Accredited by NBA (B.E - CSE, EEE, ECE, Mech & B.Tech.IT)

COIMBATORE-641 035, TAMIL NADU



DEPARTMENT OF MATHEMATICS

$$\left\{ \text{Res } f(z) \right\}_{z=2} = 0 \text{ (lies outside } c \text{)}$$

By Cauchy's residue theorem,

$$\int_c f(z) dz = 2\pi i \left[\text{Sum of residues of } f(z) \text{ at the poles which lie inside } c \right]$$

$$= 2\pi i (2 - 1)$$

$$= 2\pi i$$