



## DEPARTMENT OF MATHEMATICS

### Laurent's Series

Let  $C_1$  and  $C_2$  be two concentric circles  
 $|z-a| = R_1$  and  $|z-a| = R_2$  where  $R_2 < R_1$ .

Let  $f(z)$  be analytic inside and on the  
annular region  $R$  between  $C_1$  and  $C_2$ . Then for any  
 $z \in R$ ,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

$$\text{Where } a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{(z-a)^{n+1}} dz$$

$$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{(z-a)^{1-n}} dz$$

### Problems :

① Expand  $f(z) = \frac{z^2-1}{(z+2)(z+3)}$  in a Laurent's

series if (i)  $|z| < 2$  (ii)  $|z| > 3$  and  
(iii)  $2 < |z| < 3$ .

Soln: Using partial fractions,

$$f(z) = \frac{z^2-1}{(z+2)(z+3)} = A + \frac{B}{z+2} + \frac{C}{z+3} \rightarrow \text{①}$$

$$\frac{z^2-1}{(z+2)(z+3)} = \frac{A(z+2)(z+3) + B(z+3) + C(z+2)}{(z+2)(z+3)}$$

$$z^2-1 = A(z+2)(z+3) + B(z+3) + C(z+2)$$



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Put  $z = -2$

$$(-2)^2 - 1 = A(0) + B(-2+3) + 0$$

$$4 - 1 = B$$

$$\boxed{B = 3}$$

Put  $z = -3$

$$(-3)^2 - 1 = 0 + 0 + c(-3+2)$$

$$9 - 1 = -c$$

$$\boxed{c = -8}$$

put  $z = 0$

$$0 - 1 = A(2)(3) + B(3) + C(2)$$

$$-1 = 6A + 3B + 2C$$

$$= 6A + 3(3) + 2(-8)$$

$$= 6A + 9 - 16$$

$$-1 = 6A - 7$$

$$6A = 7 - 1 = 6$$

$$\boxed{A = 1}$$

$$\textcircled{1} \Rightarrow f(z) = 1 + \frac{3}{z+2} + \frac{8}{z+3} \rightarrow \textcircled{2}$$

(i)  $|z| < 2$

$$\Rightarrow \frac{|z|}{2} < 1$$



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$$(2) \Rightarrow f(z) = 1 + \frac{3}{2 \left(1 + \frac{z}{2}\right)} - \frac{8}{3 \left(1 + \frac{z}{3}\right)}$$

$$= 1 + \frac{3}{2} \left(1 + \frac{z}{2}\right)^{-1} - \frac{8}{3} \left(1 + \frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2} \left[1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \dots\right]$$

$$- \frac{8}{3} \left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \dots\right]$$

Formula

$$\left(\because \left(1+z\right)^{-1} = 1 - z + z^2 - z^3 + \dots\right)$$

$$(ii) |z| > 3$$

$$3 < |z|$$

$$\frac{3}{|z|} < 1$$

$$(2) \Rightarrow f(z) = 1 + \frac{3}{z \left(1 + \frac{2}{z}\right)} - \frac{8}{z \left(1 + \frac{3}{z}\right)}$$

$$= 1 + \frac{3}{z} \left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{z} \left(1 + \frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z} \left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \dots\right]$$

$$- \frac{8}{z} \left[1 - \frac{3}{z} + \left(\frac{3}{z}\right)^2 - \dots\right]$$

$$(iii) 2 < |z| < 3$$

$$2 < |z| \quad \text{and} \quad |z| < 3$$

$$\frac{2}{|z|} < 1 \quad \text{and} \quad \frac{|z|}{3} < 1$$



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$$\begin{aligned}
 \textcircled{2} \Rightarrow 1 + \frac{-3}{z \left(1 + \frac{z}{2}\right)} - \frac{8}{3 \left(1 + \frac{z}{3}\right)} & \leftarrow \textcircled{1} \\
 & = 1 + \frac{3}{z} \left(1 + \frac{z}{2}\right)^{-1} - \frac{8}{3} \left(1 + \frac{z}{3}\right)^{-1} \\
 & = 1 + \frac{3}{z} \left[1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \dots\right] \\
 & \quad - \frac{8}{3} \left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \dots\right]
 \end{aligned}$$

$e > |z|$  (i)  
 $|z| > 5$   
 $1 > \frac{|z|}{2}$   
 $1 > \frac{|z|}{3}$

$$\begin{aligned}
 \textcircled{3} \Rightarrow \frac{z}{z^2 + 1} & \leftarrow \textcircled{2} \\
 & = \frac{z}{\left(\frac{z}{2} + 1\right) \left(\frac{z}{2} - 1\right)} \\
 & = \frac{z}{\left(\frac{z}{2} + 1\right) \left(\frac{z}{2} - 1\right)} \\
 & = \frac{z}{\left(\frac{z}{2} + 1\right) \left(\frac{z}{2} - 1\right)} \\
 & = \frac{z}{\left(\frac{z}{2} + 1\right) \left(\frac{z}{2} - 1\right)}
 \end{aligned}$$

$e > |z| > 5$  (iii)  
 $e > |z|$  but  $|z| > 5$   
 $1 > \frac{|z|}{2}$  but  $1 > \frac{|z|}{3}$



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Soln:

$$\text{Given: } f(z) = \frac{7z-2}{z(z-2)(z+1)}$$

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$$

$$7z-2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$$

When  $z=0$ ,  $A=1$

$z=-1$ ,  $C=-3$

$z=2$ ,  $B=2$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

(i)  $|z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$

$$\begin{aligned} f(z) &= \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1} \\ &= \frac{1}{z} + \frac{2}{-2(1-\frac{z}{2})} - \frac{3}{z+1} \\ &= \frac{1}{z} - \left(1-\frac{z}{2}\right)^{-1} - 3(z+1)^{-1} \\ &= \frac{1}{z} - \left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots\right] - 3 \left[1 - z + z^2 - z^3 + \dots\right] \end{aligned}$$

(ii)  $|z| > 3 \Rightarrow \frac{3}{|z|} < 1$

$$\begin{aligned} f(z) &= \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1} \\ &= \left(\frac{1}{z} + \frac{2}{z(1-\frac{2}{z})}\right) - \frac{3}{z(1+\frac{1}{z})} \\ &= \frac{1}{z} + \frac{2}{z} \left(1-\frac{2}{z}\right)^{-1} - \frac{3}{z} \left(1+\frac{1}{z}\right)^{-1} \\ &= \frac{1}{z} + \frac{2}{z} \left[1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \dots\right] - \frac{3}{z} \left[1 - \frac{1}{z} + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \dots\right] \end{aligned}$$

$$(iii) \quad 2 < |z| < 3$$

$$|z| > 2, \quad |z| < 3$$

$$\Rightarrow \frac{2}{|z|} < 1, \quad \frac{|z|}{3} < 1$$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$= \frac{1}{z} + \frac{2}{z(1-\frac{2}{z})} - \frac{3}{z(1+\frac{1}{z})}$$

$$= \frac{1}{z} + \frac{2}{z} \left( 1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \dots \right) - \frac{3}{z} \left( 1 - \frac{1}{z} + \left(\frac{1}{z}\right)^2 - \dots \right)$$

$$(iv) \quad 1 < |z+1| < 3$$

$$\text{Let } u = z+1 \Rightarrow z = u-1$$

$$1 < |u| < 3$$

$$\Rightarrow \frac{1}{|u|} < 1, \quad \left| \frac{u}{3} \right| < 1$$

$$f(z) = \frac{1}{u-1} + \frac{2}{u-3} - \frac{3}{u}$$

$$= \frac{1}{u \left[ 1 - \frac{1}{u} \right]} + \frac{2}{(-3) \left( 1 - \frac{u}{3} \right)} - \frac{3}{u}$$

$$= \frac{1}{u} \left( 1 - \frac{1}{u} \right)^{-1} - \frac{2}{3} \left( 1 - \frac{u}{3} \right)^{-1} - \frac{3}{u}$$

$$= \frac{1}{u} \left( 1 + \frac{1}{u} + \left(\frac{1}{u}\right)^2 + \dots \right) - \frac{2}{3} \left( 1 + \frac{u}{3} + \left(\frac{u}{3}\right)^2 + \dots \right) - \frac{3}{u}$$

$$f(z) = \frac{1}{z+1} \left[ 1 + \left(\frac{1}{z+1}\right) + \left(\frac{1}{z+1}\right)^2 + \dots \right]$$

$$- \frac{2}{3} \left( 1 + \frac{z+1}{3} + \left(\frac{z+1}{3}\right)^2 + \dots \right) - \frac{3}{z+1}$$