Legendre's Linear Differential Equations:

$$(ax+b)^2 y'' + (ax+b)y' + y = Q(x)$$

Let $e^z = (ax+b)$

$z = \log(ax+b)$

$(ax+b)D = a\theta$

$(ax+b)^2 D^2 = a^2(\theta^2 - \theta)$

1. Transform the equation to construct coefficient

$$(2x+3)^2 y'' - (2x+3)y' + 2y = 6x$$

$$(2x+3)^2 D^2 y - (2x+3)Dy + 2y = 6x$$

$$[(2x+3)^2 D^2 - (2x+3)D + 2]y = 6x \rightarrow (1)$$

$e^z = 2x+3$ $\qquad$ $2x = e^z - 3$

$\qquad$ $x = \frac{e^z - 3}{2}$

$z = \log(2x+3)$

$(2x+3)D = 2\theta$

$(2x+3)^2 D^2 = 2^2(\theta^2 - \theta)$

$= 4(\theta^2 - \theta)$

sub in equ (1)

$$(4(\theta^2 - \theta) - 2\theta + 2)y = 6\left(\frac{e^z - 3}{2}\right)$$

$$(4\theta^2 - 4\theta - 2\theta + 2)y = 3e^z - 9$$

$$(4\theta^2 - 6\theta + 2)y = 3e^z - 9$$

2) Solve : $[(x+2)^2D^2-(x+2)D+1]y=3x+4.$

Sol:- $[(x+2)^2D^2+(x+2)D+1]y=3x+4 \rightarrow (1)$

$e^z=x+2$

$x=e^z-2$

$z=\log(x+2)$

$(x+2)D=\theta$

$(x+2)D^2=1^2(\theta^2-\theta)=\theta^2-\theta$

Sub in equ (1)

$[(\theta^2-\theta)-\theta+1]y=3[e^z-2]+4$

$(\theta^2-\theta-\theta+1)y=3e^z-6+4$

$(\theta^2-\theta(2)+1)y=3e^z-2 \longrightarrow (2)$

C.F The auxiliary equation is $m^2-2m+1=0$

$(m-1)(m-1)=0$

$m=1,1$

$C.F=(A+Bz)e^z$

$P.I_1=\frac{1}{\theta^2-2\theta+1}3e^z$

$=\frac{1}{1^2-2(1)+1}3e^z$

$=\frac{1}{0}3e^z$

$P.I_1=\frac{z}{2\theta-2}3e^z=\frac{z}{2(1)-2}3e^z=\frac{z}{0}3e^z=\frac{z}{2}3e^z$

$$P.I_2 = \frac{1}{\theta^2 - 2\theta + 1}\left(-2e^{0z}\right)$$

$$= \frac{1}{0^2 - 2(0) + 1}\left(-2(1)\right)$$

$$= \frac{1}{1}(-2)$$

$$= -2$$

$$y = C.F + PI_1 + PI_2$$

$$= (A + Bz)e^z + \frac{z^2}{2} 3e^z - 2$$

$$= \left[A + B\log(x+2)\right] e^{\log(x+2)} + \frac{3\log(x+2)^2}{2} e^{\log(x+2)} - 2$$

$$= \left[A + B\log(x+2)(x+2) + \frac{3\log(x+2)^2}{2}(x+2) - 2\right]$$

$$\boxed{y = \left[A + B\log(x+2)(x+2) + \frac{3\log(x+2)^2}{2}(x+2) - 2\right]}$$

2) Solve: $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\log(1+x)$.

Sol: $(1+x)^2 D^2 + (1+x)D + y = 4\cos\log(1+x) \longrightarrow (1)$

$e^z = 1+x$

$x = e^z - 1$

$z = \log(1+x)$

$(1+x)D = \theta$

$(1+x)D^2 = 1^2(\theta^2-\theta) = \theta^2-\theta$

Sub in equ (1)

$(\theta^2-\theta)+\theta+1)y = 4\cos\log(1+(e^z-1))$

$(\theta^2-\theta+\theta+1)y = 4\cos\log(1+e^z-1)$

$(\theta^2+1)y = 4\cos\log(e^z) \longrightarrow (2)$

The auxiliary equ is $m^2+1=0$

$m^2 = -1$

$m = \pm i = 0 \pm i$

$C.F = e^{\alpha x}(A\cos\beta x + B\sin\beta x) = e^{0x}(A\cos x + B\sin x)$

$C.F = A\cos x + B\sin x$.

$P.I = \frac{1}{\theta^2+1} 4\cos\log(e^z)$

$= \frac{1}{\theta^2+1} 4\cos\log(e^z)$

$$= \frac{1}{-1+1} 4\cos\log(e^2) \qquad z \to -1$$

$$= \frac{1}{0} 4\cos\log(e^2)$$

$$= \frac{z}{2\theta} 4\cos\log(e^2)$$

$$= \frac{z}{2} 4\cos\log(e^2)$$

$$= 2z\sin\log(1+x)$$

$$y = A\cos x + B\sin x + 2z\sin\log(1+x).$$

$$= A\cos x + B\sin x + 2\log(1+x)\sin\log(1+x)$$

Euler type of Homogeneous Equation:-

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = f(x) \quad \to (1)$$

$a_1, a_2, \ldots a_n$ are constants and $f(x)$ is a function of x is called a linear differential equation with variable coefficients.

Equ (1) can be reduced to a linear differential equation with constant coefficients by

$$x = e^z \text{ (or) } z = \log x$$

$$\frac{dy}{dx} = \frac{1}{x} \cdot \frac{dy}{dz}$$