Homogeneous type on Legendre's differential Equations

$$(ax+b)^2y'' + (ax+b)y' + y = Q(x)$$

Let $e^z = (ax+b)$

$z = \log(ax+b)$

$(ax+b)D = a\theta, \quad (ax+b)^2D^2 = a^2(\theta^2-\theta)$

Problems:

1) Solve : $[(x+2)^2D^2 - (x+2)D+1]y = 3x+4$

Sol: $[(x+2)^2D^2-(x+2)D+1]y = 3x+4 \longrightarrow (1)$

$e^z = x+2$

$x = e^z-2$

$z = \log(x+2)$

$(x+2)D = \theta, \quad (x+2)^2D^2 = 1^2(\theta^2-\theta) = \theta^2-\theta$

Sub in equ (1), $((\theta^2-\theta)-\theta+1)y = 3(e^z-2)+4$

$(\theta^2-\theta-\theta+1)y = 3e^z-6+4$

$(\theta^2-2\theta+1)y = 3e^z-2 \longrightarrow (2)$

C.F $\Rightarrow$ $m^2-2m+1=0 \Rightarrow (m-1)(m-1)=0$

$m = 1, 1$

$C.F = (A+Bz)e^z$

$P.I_1 = \dfrac{1}{\theta^2-2\theta+1} 3e^z$

$= \dfrac{1}{1^2-2(1)+1} 3e^z$

$= \dfrac{1}{0} 3e^z$

$= \dfrac{z}{2\theta-2} 3e^z$

$= \dfrac{z}{2(1)-2} 3e^z$

$= \dfrac{z}{0} 3e^z$

$= \dfrac{z^2}{2} 3e^z$

$P.I_2 = \dfrac{1}{\theta^2-2\theta+1}(-2e^{0z})$

$= \dfrac{1}{\theta^2-2(0)+1}(-2(1))$

$= \dfrac{1}{1}(-2)$

$P.I_2 = -2$

$y = C.F + P.I_1 + P.I_2$

$y = (A+Bz)e^z + \frac{z^2}{2} 3e^z - 2$

$= [A + B\log(x+2)] e^{\log(x+2)} + \frac{3[\log(x+2)]^2}{2} e^{\log(x+2)} - 2$

$= [A + B\log(x+2)(x+2) + \frac{3(\log(x+2))^2}{2}(x+2) - 2$

$$\boxed{y = [A + B\log(x+2)(x+2) + \frac{3(\log(x+2))^2}{2}(x+2) - 2]}$$

2) Solve: $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\log(1+x)$.

Sol: $(1+x)^2 D^2 + (1+x)D + y = 4\cos\log(1+x) \longrightarrow (1)$

$e^z = 1+x, \quad x = e^z - 1, \quad z = \log(1+x)$

$(1+x)D = \theta, \quad (1+x)^2 D^2 = 1^2(\theta^2 - \theta) = \theta^2 - \theta$

sub in equ (1)

$(\theta^2 - \theta) + \theta + 1)y = 4\cos\log(\cancel{1} + (e^z - \cancel{1}))$

$(\theta^2 - \theta + \theta + 1)y = 4\cos\log e^z \rightarrow (2)$

The auxiliary equ is $m^2 + 1 = 0$

$m^2 = -1$

$m = \pm i \Rightarrow C.F = (A\cos\beta x + B\sin\beta x)e^{\alpha x}$

$C.F = A\cos x + B\sin x$

$P.I = \frac{1}{\theta^2+1} 4\cos\log(e^z)$

$= \frac{1}{-1+1} 4\cos\log(e^z)$

$= \frac{1}{0} 4\cos\log(e^z)$

$= \frac{z}{2\theta} 4\cos\log(e^z)$

$= \frac{z}{\cancel{2}} \cancel{4}^{2} \cos\log(e^z)$

$= 2z\sin\log(e^z)$

$y = C.F + P.I$

$$\boxed{y = A\cos x + B\sin x + 2\log(1+x) \sin\log(1+x).}$$

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Legendre's type

4) Solve:. $(2x+3)^2 \dfrac{d^2y}{dx^2} - (2x+3)\dfrac{dy}{dx} - 12y = 6x.$

Sol:. The given equation can be

$$\left((2x+3)^2D^2 - (2x+3)D - 12\right)y = 6x$$

$$2x+3 = e^z \quad , z = \log(2x+3) \qquad D' = \frac{d}{dz}$$

$$(2x+3)D = 2D' , \quad (2x+3)^2D^2 = 2^2D'(D'-1)$$

$$[4D'(D'-1) - 2D' - 12]y = 3(e^z - 3)$$

$$(4D'^2 - 6D' - 12)y = 3e^z - 9$$

The auxiliary equ^n is $4m^2 - 6m - 12 = 0$

$$m = \frac{6 \pm \sqrt{36+192}}{8}$$

$$m = \frac{6 \pm 2\sqrt{57}}{8}$$

$$m = \frac{3 \pm \sqrt{57}}{4}$$

$$C.F = C_1 e^{m_1 z} + C_2 e^{m_2 z}$$

$$= C_1 e^{\frac{3+\sqrt{57}}{4}z} + C_2 e^{\frac{3-\sqrt{57}}{4}z}$$

$$P.I = \frac{1}{4D'^2 - 6D' - 12}\left(3e^z - 9e^{0z}\right)$$

$$= \frac{1}{4-6-12}\, 3e^z - \frac{9}{-12}$$

$$= \frac{-3}{14}e^z + \frac{3}{4}$$

$$y = C.F + P.I$$

$$= C_1 e^{m_1 z} + C_2 e^{m_2 z} - \frac{3}{14}e^z + \frac{3}{4}$$

$$y = C_1 e^{\frac{3+\sqrt{57}}{4}\log(1+x)} + C_2 e^{\frac{3-\sqrt{57}}{4}\log(1+x)} - \frac{3}{14} e^{\log(2x+3)} + \frac{3}{4} \;//$$

3) Solve:- $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=2\sin[\log(1+x)]$.

Ans:- $y=C_1\cos z+C_2\sin z-z\cos z$

where $z=\log(1+x)$.

Euler type

An equation of the form $x^n\frac{d^ny}{dx^n}+a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}}+\cdots$

$\cdots+a_ny=f(x)$ $\rightarrow$ (1)

$a_1, a_2, \ldots a_n \Rightarrow$ Constants

$f(x) \Rightarrow$ function of x.

$x=e^z, \quad z=\log x$

$x\frac{dy}{dz}=\frac{dy}{dz}=D'y$

$x^2\frac{d^2y}{dz^2}=D'(D'-1)y$

$x^3\frac{d^3y}{dx^3}=D'(D'-1)(D'-2)y$

$x^4\frac{d^4y}{dx^4}=D'(D'-1)(D'-2)(D'-3)y$

Problems:

1) Solve: $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-9y=10+\frac{5}{x^2}$.

Sol: $(x^2D^2+xD-9)y=10+\frac{5}{x^2}$.

$x=e^z, \quad z=\log x$.

$xD=D', \quad x^2D^2=D'(D'-1) \rightarrow$ (2)

$\frac{1}{x^2}=\frac{1}{(e^z)^{2}}=\frac{1}{e^{2z}}=e^{-2z}$

Sub (2) in (1) equ, we get

$[D'(D'-1)+D'-9]y=10+5e^{-2z}$

$(D'^2-9)y=10+5e^{-2z}$

The auxiliary equ is $m^2-9=0$

$m=\pm 3$

$C.F=Ae^{3z}+Be^{-3z}$