3) Solve: $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin[\log(1+x)]$.

Ans: $y = C_1\cos z + C_2\sin z - z\cos z$

where $z = \log(1+x)$.

Euler type

An equation of the form $x^n\frac{d^ny}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_ny = f(x)$ ——— (1)

$a_1, a_2, \ldots a_n \Rightarrow$ Constants

$f(x) \Rightarrow$ function of x.

$x = e^z$, $z = \log x$

$x\frac{dy}{dz} = \frac{dy}{dz} = D'y$

$x^2\frac{d^2y}{dz^2} = D'(D'-1)y$

$x^3\frac{d^3y}{dx^3} = D'(D'-1)(D'-2)y$

$x^4\frac{d^4y}{dx^4} = D'(D'-1)(D'-2)(D'-3)y$

Problems:

1) Solve: $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 9y = 10 + \frac{5}{x^2}$.

Sol: $(x^2D^2 + xD - 9)y = 10 + \frac{5}{x^2}$.

$x = e^z$, $z = \log x$.

$xD = D'$, $x^2D^2 = D'(D'-1)$ ——— (2)

$\frac{1}{x^2} = \frac{1}{(e^z)^2} = \frac{1}{e^{2z}} = e^{-2z}$

Sub (2) in (1) eqn, we get

$[D'(D'-1) + D' - 9]y = 10 + 5e^{-2z}$

$(D'^2 - 9)y = 10 + 5e^{-2z}$

The auxiliary equ is $m^2 - 9 = 0$

$m = \pm 3$

$C.F = Ae^{3z} + Be^{-3z}$

$$P.I = \frac{1}{D'^2 - 9}(10 + 5e^{-2z})$$

$$= \frac{1}{D'^2 - 9} 10e^{0z} + \frac{5}{D'^2 - 9} e^{-2z}$$

$$= \frac{1}{0 - 9} 10e^{0z} + \frac{5}{(-2)^2 - 9} e^{-2z}$$

$$P.I = \frac{-10}{9} - e^{-2z}$$

$$y = C.F + P.I$$

$$y = Ae^{3z} + Be^{-3z} - \frac{10}{9} e^{-2z}$$

$$= Ae^{3\log x} + Be^{-3\log x} - \frac{10}{9} e^{-2\log x}$$

$$= Ax^3 + Bx^{-3} - \frac{10}{9} - x^{-2}$$

$$y = Ax^3 + \frac{B}{x^3} - \frac{10}{9} - \frac{1}{x^2}$$

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2) Solve: $x^2y'' - 3xy' - 5y = \cos(\log x)$

Sol:- $(x^2D^2 - 3xD - 5)y = \cos(\log x) \rightarrow (1)$

$x = e^z$, $z = \log x$

$xD = D'$, $x^2D^2 = D'(D' - 1) \rightarrow (2)$

Sub (2) in (1), we get

$$(D'(D' - 1) - 3D' - 5)y = \cos z$$

$$(D'^2 - 4D' - 5)y = \cos z$$

The auxiliary equ is $m^2 - 4m - 5 = 0$

$m = 5, m = -1.$

$$C.F = Ae^{-z} + Be^{5z}$$

$$P.I = \frac{1}{D'^2 - 4D' - 5} \cos z$$

$$= \frac{1}{-1-4D'-5}\cos z$$

$$= \frac{1}{-4D'-6}\cos z$$

$$= \frac{-4D'+6}{(-4D'+6)(-4D'-6)}\cos z$$

$$= -\frac{(4D'-6)}{16D'^2-36}\cos z$$

$$= -\frac{(-4\sin z-6\cos z)}{-52}$$

$$= \frac{-1}{52}(4\sin z+6\cos z)$$

$$P.I = \frac{-1}{26}(2\sin z+3\cos z)$$

$y = C.F + P.I$

$$y = Ae^{-z} + Be^{5z} - \frac{1}{26}(2\sin z+3\cos z)$$

$$= Ae^{-\log x} + Be^{5\log x} - \frac{1}{26}(2\sin\log x+3\cos\log x)$$

$$y = \frac{A}{x} + Bx^5 - \frac{1}{26}(2\sin(\log x)+3\cos(\log x))$$

$e^{\log x} = x$
$z = \log x$

3) Solve :: $(x^2D^2+xD-9)y = x^2\log x$

Ans: $Ax^3 + \frac{B}{x^3} - \frac{x^2}{25}(5\log x+4)$

4) Solve : $(x^2D^2-3xD)y = x+11$

Ans :- $y = A+Bx^4 - \frac{x}{3} - \frac{11}{4}\log x$

5) Solve :- $x^2\frac{d^2y}{dx^2} - 7x\frac{dy}{dx} + 12y = x^2$

Ans : $Ax^2+Bx^6 - \frac{x^2\log x}{4}$, $x = e^z$