Legendre's Linear Differential Equations:-

$$(ax+b)^2 y'' + (ax+b)y' + y = Q(x)$$

Let $e^z = (ax+b)$

$$z = \log(ax+b)$$

$$(ax+b)D = a\theta$$

$$(ax+b)^2 D^2 = a^2(\theta^2-\theta)$$

1. Transform the equation to construct coefficient

$$(2x+3)^2 y'' - (2x+3)y' + 2y = 6x$$

$$(2x+3)^2 D^2 y - (2x+3)Dy + 2y = 6x$$

$$[(2x+3)^2 D^2 - (2x+3)D + 2]y = 6x \longrightarrow (1)$$

$e^z = 2x+3$ $\qquad$ $2x = e^z - 3$

$z = \log(2x+3)$ $\qquad$ $x = \dfrac{e^z-3}{2}$

$$(2x+3)D = 2\theta$$

$$(2x+3)^2 D^2 = 2^2(\theta^2-\theta)$$

$$= 4(\theta^2-\theta)$$

sub in equ (1)

$$(4(\theta^2-\theta) - 2\theta + 2)y = 6\left(\frac{e^z-3}{2}\right)$$

$$(4\theta^2 - 4\theta - 2\theta + 2)y = 3e^z - 9$$

$$(4\theta^2 - 6\theta + 2)y = 3e^z - 9$$

2) Solve : $[(x+2)^2D^2-(x+2)D+1]y = 3x+4.$

Sol:- $[(x+2)^2D^2+(x+2)D+1]y = 3x+4 \longrightarrow (1)$

$e^z = x+2$

$x = e^z - 2$

$z = \log(x+2)$

$(x+2)D = \theta$

$(x+2)D^2 = 1^2(\theta^2-\theta) = \theta^2-\theta$

Sub in equ (1)

$[(\theta^2-\theta)-\theta+1]y = 3[e^z-2]+4$

$(\theta^2-\theta-\theta+1)y = 3e^z-6+4$

$(\theta^2-\theta(2)+1)y = 3e^z-2 \longrightarrow (2)$

C.F The auxiliary equation is $m^2-2m+1=0$

$(m-1)(m-1)=0$

$m = 1,1$

$C.F = (A+Bz)e^z$

$P.I_1 = \dfrac{1}{\theta^2-2\theta+1}\,3e^z$

$= \dfrac{1}{1^2-2(1)+1}\,3e^z$

$= \dfrac{1}{0}\,3e^z$

$P.I_1 = \dfrac{z}{2\theta-2}\,3e^z = \dfrac{z}{2(1)-2}\,3e^z = \dfrac{z}{0}\,3e^z = \dfrac{z}{2}\,3e^z$

2) Solve: $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\log(1+x)$.

Sol: $(1+x)^2 D^2 + (1+x) D + y = 4\cos\log(1+x) \longrightarrow (1)$

$e^z = 1+x$

$x = e^z - 1$

$z = \log(1+x)$

$(1+x) D = \theta$

$(1+x) D^2 = 1^2(\theta^2 - \theta) = \theta^2 - \theta$

Sub in equ (1)

$(\theta^2 - \theta) + \theta + 1)y = 4\cos\log(1+(e^z-1))$

$(\theta^2 - \theta + \theta + 1)y = 4\cos\log(\cancel{1}+e^z\cancel{-1})$

$(\theta^2+1)y = 4\cos\log(e^z) \longrightarrow (2)$

The auxiliary equ is $m^2+1=0$

$m^2 = -1$

$m = \pm i = 0 \pm i$

$C.F = e^{\alpha x}(A\cos\beta x + B\sin\beta x) = e^{0x}(A\cos x + B\sin x)$

$C.F = A\cos x + B\sin x.$

$P.I = \frac{1}{\theta^2+1} 4\cos\log(e^z)$

$= \frac{1}{\theta^2+1} \cancel{4\cos(e^z)}\ 4\cos\log(e^z)$

$$P.I_2 = \frac{1}{\theta^2 - 2\theta + 1}\left(-2e^{0z}\right)$$

$$= \frac{1}{0^2 - 2(0) + 1}\left(-2(1)\right)$$

$$= \frac{1}{1}(-2)$$

$$= -2$$

$$y = C.F + PI_1 + PI_2$$

$$= (A+Bz)e^{z} + \frac{z}{2}\,3e^{z} - 2$$

$$= \left[A + B\log(x+2)\right]e^{\log(x+2)} + \frac{3\log(x+2)^2}{2}e^{\log(x+2)} - 2$$

$$= \left[A + B\log(x+2)(x+2) + 3\frac{\log(x+2)^2}{2}(x+2) - 2\right]$$

$$\boxed{y = \left[A + B\log(x+2)(x+2) + \frac{3\log(x+2)^2}{2}(x+2) - 2\right]}$$

$$x\frac{dy}{dz} = \frac{dy}{dz} \quad \text{(or)} \quad x\frac{dy}{dx} = D'y$$

$$D' = \frac{d}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$= \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\cdot\frac{d^2y}{dz^2}\cdot\frac{1}{x}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$$= (D'^2 - D')y \quad \text{where } D' = \frac{d}{dz}$$

$$x^2\frac{d^2y}{dx^2} = D'(D'-1)y$$

$|||^{ly}$, $$x^3\frac{d^3y}{dx^3} = D'(D'-1)(D'-2)y$$

$$x^4\frac{d^4y}{dx^4} = D'(D'-1)(D'-2)(D'-3)y$$

1) Solve: $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 9y = 10 + \dfrac{5}{x^2}$

Sol: $(x^2D^2 + xD - 9)y = 10 + \dfrac{5}{x^2}$

$x = e^z$

$z = \log x$

$$\left.\begin{aligned} xD &= D' \\ x^2D^2 &= D'(D'-1), \quad D' = \frac{d}{dz} \end{aligned}\right\} \rightarrow (2)$$

Sub (2) in (1),

$$[D'(D'-1) + D' - 9]y = 10 + 5e^{-2z}$$

$$(D'^2 - 9)y = 10 + 5e^{-2z}$$

The auxiliary equ is $m^2 - 9 = 0$

$m = \pm 3$

$$C.F = Ae^{3z} + Be^{-3z}$$

$$P.I = \frac{1}{D'^2 - 9}(10 + 5e^{-2z})$$

$$= \frac{1}{D'^2-9} 10e^{0x} + \frac{5}{D'^2-9} e^{-2z}$$

$$= \frac{1}{0-9} 10 + \frac{5}{(-2)^2-9} e^{-2z}$$

$$P.I = \frac{-10}{9} - e^{-2z}$$

$$= \frac{1}{-1-4D'-5} \cos z$$

$$= \frac{1}{-4D'-6} \cos z$$

$$= -\frac{(4D'-6)}{(4D'+6)(4D'-6)} \cos z$$

$$= -\frac{(4D'-6)}{16D'^2-36} \cos z \qquad D'^2 \to -1$$

$$= -\frac{(4D'-6)\cos z}{16(-1)-36}$$

$$= -\frac{(-4\sin z - 6\cos z)}{-52}$$

$$= -\frac{1}{52}(4\sin z + 6\cos z)$$

$$P.I = -\frac{1}{26}(2\sin z + 3\cos z)$$

$$y = C.F + P.I$$

$$y = Ae^{-z} + Be^{5z} - \frac{1}{26}(2\sin z + 3\cos z)$$

$$= Ae^{-\log x} + Be^{5\log x} - \frac{1}{26}(2\sin \log x + 3\cos \log x)$$

$$= \frac{A}{x} + Bx^5 - \frac{1}{26}(2\sin(\log x) + 3\cos(\log x))$$

$$\boxed{e^{\log x} = x,\ z = \log x}$$

$$= \frac{1}{-1+1} 4\cos\log(e^2) \qquad z \to -1$$

$$= \frac{1}{0} 4\cos\log(e^2)$$

$$= \frac{z}{2\theta} 4\cos\log(e^2)$$

$$= \frac{\cancel{z}}{\cancel{z}} \overset{2}{\cancel{4}}\cos\log(e^2)$$

$$= 2z\sin\log(1+x)$$

$$y = A\cos x + B\sin x + 2z\sin\log(1+x).$$

$$= A\cos x + B\sin x + 2\log(1+x)\sin\log(1+x)$$

Homogeneous Equation: