First Shifting property:

If $L\{f(t)\} = F(s)$ then

(i) $L[e^{-at} f(t)] = \{L[f(t)]\}_{s \to s+a} = F(s+a)$

(ii) $L[e^{at} f(t)] = \{L[f(t)]\}_{s \to s-a} = F(s-a)$

Proof:

(i) We know that,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

$$L[e^{-at} f(t)] = \int_0^\infty e^{-st} [e^{-at} f(t)]\, dt$$

$$= \int_0^\infty e^{-(s+a)t} f(t)\, dt$$

$$= F(s+a)$$

(ii) $$L[e^{at} f(t)] = \int_0^\infty e^{-st} [e^{at} f(t)]\, dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt$$

$$= F(s-a).$$

Second Shifting property:

If $L\{f(t)\} = F(s)$ and $g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$

then $L[g(t)] = e^{-as} F(s)$.

Proof:

$$L[g(t)] = \int_0^\infty e^{-st} g(t)\, dt$$

$$= \int_0^a e^{-st} g(t)\, dt + \int_a^\infty e^{-st} g(t)\, dt$$

$$L[g(t)] = 0 + \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

Put $t-a = u \Rightarrow dt = du$

When $t = a \Rightarrow u = 0$

$t \to \infty \Rightarrow u \to \infty$

$$L[g(t)] = \int_0^\infty e^{-s(u+a)} f(u)\,du$$

$$= \int_0^\infty e^{-us} e^{-as} f(u)\,du$$

$$= e^{-as} \int_0^\infty e^{-us} f(u)\,du$$

$$= e^{-as} \int_0^\infty e^{-st} f(t)\,dt \quad \text{Replace } u \to t$$

$$L[g(t)] = e^{-as} F(s).$$

Laplace transforms of derivatives:

If $L[f(t)] = F(s)$ then

$$L[f'(t)] = s F(s) - f(0).$$

Proof:

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt$$

Integrating by parts we get,

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty f(t)(-se^{-st})\,dt$$

$$= \left[e^{-\infty} f(\infty) - e^{0} f(0) \right] + s \int_0^\infty e^{-st} f(t)\, dt$$

$$= -f(0) + s\, L\{f(t)\}$$

$$= s\,F(s) - f(0).$$

Corollary:

Let $f''(t) = s^2 F(s) - s f(0) - f'(0)$

Let $L[g'(t)] = s\, G(s) - g(0)$.

We know that,

$$L[f'(t)] = s\, L[f(t)] - f(0).$$

Replace $f(t) \to f'(t)$ & $f'(t) \to f''(t)$ &

$f(0) \to f'(0)$.

$$\Rightarrow L[f''(t)] = s\, L[f'(t)] - f'(0)$$

$$= s\left[s\left(L[f(t)] - f(0) \right) \right] - f'(0)$$

$$= s^2 L[f(t)] - s f(0) - f'(0)$$

$$= s^2 F(s) - s f(0) - f'(0).$$

Laplace Transform of integrals:

If $L[f(t)] = F(s)$ then $L\left[\int_0^t f(t)\, dt \right] = \frac{F(s)}{s}$

Proof:

Let $g(t) = \int_0^t f(t)\, dt$ and $g(0) = 0$

then $g'(t) = f(t)$

WKT $L[g'(t)] = s\, L(g(t)) - g(0)$

$$= s\, L(g(t))$$

$$\Rightarrow L[g(t)] = \frac{1}{s} L[g'(t)]$$

$\Rightarrow L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s} L[f(t)]$ $\quad \left\{ \because g(t) = \int_0^t f(t) \right.$, $g'(t) = f(t)$

$\Rightarrow L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$

Derivative of Laplace Transform (or) Laplace transform of $t f(t)$:

If $L[f(t)] = F(s)$ then

$L[t f(t)] = -\frac{d}{ds} F(s)$

Proof:

We know that,

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\frac{d}{ds} F(s) = \frac{d}{ds} \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty \frac{\partial}{\partial s}(e^{-st}) f(t)\,dt$$

$$= \int_0^\infty -t e^{-st} f(t)\,dt$$

$$= -\int_0^\infty e^{-st}\, t f(t)\,dt$$

$$= -L[t f(t)]$$

$$\Rightarrow L[t f(t)] = -\frac{d}{ds}[F(s)]$$

In general,

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

Problems :

Change of Scale property :

(1) Find $L[\sinh 3t]$ by using change of scale property

Soln:

$$L[\sinh t] = \frac{1}{s^2-1} = F(s)$$

$$L[\sinh 3t] = \frac{1}{3} F\left(\frac{s}{3}\right)$$

$$= \frac{1}{3} \frac{1}{\left(\frac{s}{3}\right)^2 - 1}$$

$$= \frac{1}{3}\left(\frac{9}{s^2-9}\right)$$

$$= \frac{3}{s^2-9}$$

(2) Find $L(\cos 5t)$ using change of scale property?

Soln:

$$L(\cos t) = \frac{s}{s^2+1} = F(s)$$

$$L(\cos 5t) = \frac{1}{5} F\left(\frac{s}{5}\right)$$

$$= \frac{1}{5}\left[\frac{s/5}{\left(\frac{s}{5}\right)^2+1}\right]$$

$$= \frac{1}{5}\left[\frac{5s}{s^2+25}\right]$$

$$= \frac{s}{s^2+25}$$