Example:

Verify green's theorem for $\int (xy + y^2) \, dx + x^2 \, dy$ where c is the closed region bounded by $y = x^2$, $y = x$

Soln:

Given,

$$\int_C (xy + y^2)\, dx + x^2\, dy$$

$$y = x^2 \rightarrow \textcircled{1}$$
$$y = x \rightarrow \textcircled{2}$$

sub $y = x$ in $\textcircled{1}$

$$x = x^2$$

$$x^2 - x = 0$$
$$x(x-1) = 0$$
$$x = 0 \cup x - 1 = 0$$
$$x = 0 \qquad x = 1$$

If $x = 0$ then $y = 0$
If $x = 1$ then $y = 1$

$$\frac{\partial M}{\partial y} = 2y + x \qquad \frac{\partial N}{\partial x} = 2x$$

RHS : $= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$

$$= \int_0^1 \int_{x^2}^x (2x - x - 2y)\, dx\, dy$$

$$= \int_0^1 \int_{x^2}^x (x - 2y)\, dy\, dx$$

$$= \int_0^1 \left[\int_{x^2}^x x\, dy - \int_{x^2}^x 2y\, dy \right] dx$$

$$= \int_0^1 \left[(xy)_{x^2}^x - 2 \left[\frac{y^2}{2} \right]_{x^2}^x \right] dx$$

$$= \int_0^1 \left(x^2 - x^3 - 2 \left[\frac{x^2}{2} - \frac{x^4}{2} \right] \right) dx$$

$$= \int_0^1 \left[x^2 - x^3 - 2 \left[\frac{x^2 - x^4}{2} \right] \right] dx$$

$$= \int_0^1 (x^2 - x^3 - x^2 + x^4)\, dx$$

$$= \int_0^1 (x^4 - x^3)\, dx$$

$$= \int_0^1 x^4 \, dx - \int_0^1 x^3 \, dx$$

$$= \left[\frac{x^5}{5}\right]_0^1 - \left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{5} - \frac{1}{4} = \frac{4-5}{20}$$

$$= \frac{-1}{20}.$$

__LHS :__

$$\int_C (xy+y^2)\,dx + x^2\,dy = \int_{OA} (xy+y^2)\,dx + x^2\,dy + \int_{AO} (xy+y^2)\,dx + x^2\,dy$$

Along OA, $x^2 = y$

$$2x = \frac{dy}{dx}$$

$$dy = 2x\,dx$$

$$= \int_0^1 (x\cdot x^2 + x^4)\,dx + x^2\,(2x)\,dx$$

$$= \int_0^1 (x^3 + x^4)\,dx + 2x^3\,dx$$

$$= \int_0^1 (x^3 + x^4)\,dx + \int_0^1 2x^3\,dx$$

$$= \left[\left[\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1 + \left[\frac{2x^4}{4}\right]_0^1\right]$$

$$= \left[\frac{1}{4} + \frac{1}{5}\right] + \frac{1}{2}$$

$$= \frac{9}{20} + \frac{1}{2} = \frac{19}{20}.$$

Along AO , $x = y$

$$1 = \frac{dy}{dx}$$

$$\int_{AO} (xy + y^2)\, dx + x^2\, dy$$

$$= \int_1^0 (x^2 + x^2)\, dx + x^2\, dx$$

$$= \int_1^0 (x^2 + x^2 + x^2)\, dx$$

$$= \int_1^0 3x^2\, dx$$

$$= \left[\frac{3x^3}{3}\right]_1^0$$

$$= -1$$

Now,

$$LHS = \int (xy + y^2)\, dx + x^2\, dy + \int (xy + y^2)\, dx + x^2\, dy$$

$$= \frac{19}{20} + (-1)$$

$$= \frac{19}{20}$$

$$\therefore LHS = RHS$$

$$\therefore \text{Hence proved.}$$