



Harmonic function:

An expression of the form $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ is called the Laplace Equation in two dimension.

Any function having continuous second order partial derivatives which satisfies the Laplace equation is called harmonic function.

Any two harmonic functions u and v such that $f(z) = u + iv$ is analytic are called conjugate harmonic functions.

Note:

Both real and imaginary parts of an analytic function are harmonic but the converse need not to be true.

Problems:

1) Give an example such that u and v are harmonic but $u + iv$ is not analytic

sol:

$$\text{Let } w = \bar{z}$$

$$u + iv = x - iy$$

$$u = x, v = -y$$

$$\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = -1, \frac{\partial^2 u}{\partial x^2} = 0,$$

$$\frac{\partial^2 u}{\partial y^2} = 0, \frac{\partial^2 v}{\partial x^2} = 0, \frac{\partial^2 v}{\partial y^2} = 0$$



To Prove: $v-iu$ is also analytic.

we have to Prove $\frac{\partial v}{\partial x} = \frac{\partial(-u)}{\partial y}$ & $\frac{\partial v}{\partial y} = -\frac{\partial(-u)}{\partial x}$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \& \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

we know that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

Hence, $v-iu$ is also analytic.

b) If $w = e^z$ find $\frac{dw}{dz}$ using complex variable.

Sol: $w = e^z$

$$u+iv = e^{x+iy} = e^x \cdot e^{iy} = e^x (\cos y + i \sin y)$$

$$= e^x \cos y + i e^x \sin y$$

$$u = e^x \cos y \quad v = e^x \sin y$$

$$u_x = e^x \cos y \quad v_x = e^x \sin y$$

[Result: If $w = f(z) = u+iv$ then $\frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$

finding $\frac{dw}{dz}$ in terms of partial derivatives w.r to z

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = e^x \cos y + i e^x \sin y$$

$$= e^x (\cos y + i \sin y)$$

$$= e^x e^{iy} = e^{x+iy} = e^z$$



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Problems:-

1) Show that the function $u = \frac{1}{2} \log(x^2 + y^2)$ is harmonic and find its harmonic conjugate.

Sol:- Let $u = \frac{1}{2} \log(x^2 + y^2)$

$$u_x = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} \cdot 2x = \frac{x}{x^2 + y^2}$$

$$u_{xx} = \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$u_y = \frac{1}{2} \cdot \frac{1}{x^2 + y^2} \cdot 2y = \frac{y}{x^2 + y^2}$$

$$u_{yy} = \frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\therefore u_{xx} + u_{yy} = 0$$

u satisfies the Laplace equation.

u is harmonic.

$$v = \int \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) + c$$

$$= \int \left(-\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \right) + c$$

$$= \int \frac{xdy - ydx}{x^2 + y^2} + c$$



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$$= \frac{6xy^2 - 2y^3}{(x^2 + y^2)^3}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{2y^3 - 6xy^2}{(x^2 + y^2)^3} + \frac{6xy^2 - 2y^3}{(x^2 + y^2)^3} = 0$$

$\therefore v$ is a harmonic function

$$\text{But } \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} ; \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

$f(z) = u + iv$ is not analytic (or) not regular function.

Construction of Conjugate harmonic function ::

Method : 1

Suppose u is given, then

$$v = \int \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) + c, \text{ where } c \text{ is a}$$

Constant.

Method : 2

Suppose v is given, then

$$u = \int \left(\frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy \right) + c, \text{ where } c \text{ is a}$$

Constant.



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$\therefore u$ is a harmonic function.

$$\text{Let } v = \frac{-y}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{-[(x^2+y^2)(0) - y(2x)]}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)^2(2y) - 2xy \cdot 2(x^2+y^2)(2x)}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)^2(2y) - 8x^2y(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2) - 8x^2y}{(x^2+y^2)^3}$$

$$= \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = \frac{-[(x^2+y^2) - y \cdot 2y]}{(x^2+y^2)^2} = \frac{-(x^2-y^2)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)^2(2y) - (y^2-x^2)2(x^2+y^2)2y}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)^2(2y) - 4y(y^2-x^2)(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)(2y) - 4y(y^2-x^2)}{(x^2+y^2)^3}$$

$$u_x = v_y \\ -v_x = u_y$$



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$$= \int \frac{x dy - y dx}{x^2 (1 + y^2/x^2)}$$

$$= \int \frac{d(y/x)}{1 + y^2/x^2} + c$$

$$v = \tan^{-1}(y/x) + c$$

2) Prove that the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic find the conjugate harmonic function.

Sol: Let $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$u_x = 3x^2 - 3y^2 + 6x$$

$$u_y = -6xy - 6y$$

$$u_{xx} = 6x + 6$$

$$u_{yy} = -6x - 6$$
$$= -6(x+1)$$

$$\therefore u_{xx} + u_{yy} = 0$$

$\therefore u$ satisfies the Laplace equation.

$\therefore u$ is harmonic

$$\text{Now, } v = \int \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) + c$$

$$= \int (6xy + 6y) dx + (3x^2 - 3y^2 + 6x) dy + c$$

$$= \frac{6x^2y}{2} + 6xy + 3x^2y - \frac{3y^3}{3} + 6xy + c$$

$$= \frac{1}{2} [6x^2y + 12xy + 6x^2y - 2y^3 + 12xy + 2c]$$

$$v = 6x^2y + 12xy - y^3 + c$$



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3) show that $u = \cos x \cosh y$ is harmonic and hence find its harmonic conjugate.

Sol: $u = \cos x \cosh y$

$$u_x = -\sin x \cosh y$$

$$u_y = \cos x \sinh y$$

$$u_{xx} = -\cos x \cosh y$$

$$u_{yy} = \cos x \cosh y$$

$$u_{xx} + u_{yy} = 0$$

$\therefore u$ satisfies the Laplace equation.

$\therefore u$ is harmonic.

Now, $v = \int \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) + c$

$$= \int (-\cos x \sinh y dx) + (-\sin x \cosh y) dy + c$$

$$= -\sin x \sinh y - \sin x \sinh y + c$$

$$v = -2\sin x \sinh y + c.$$

Conjugate Harmonic:

If $f(z) = u + iv$ is an analytic function of z then u and v are harmonic functions and also v is the harmonic conjugate of u and u is the harmonic conjugate of v .

Conjugate of v :

To find harmonic conjugate: If u is given (Real part)

$$v = \int \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) + c.$$



Here, $\frac{\partial u}{\partial x}$ is the terms not involving 'x'.

$$v = \int \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) + c.$$

If v is given (imaginary part)

$$u = \int -\frac{\partial v}{\partial y} dx + \frac{\partial v}{\partial x} dy + c.$$

1) Verify $u = 2x - x^3 + 3xy^2$ is harmonic and determine its harmonic conjugate.

Sol: Given $u = 2x - x^3 + 3xy^2$

Condition is $u_{xx} + v_{yy} = 0$

$$u_x = 2 - 3x^2 + 3y^2$$

$$u_y = 6xy$$

$$u_{xx} = -6x$$

$$u_{yy} = 6x$$

$$u_{xx} + v_{yy} = -6x + 6x = 0$$

$\therefore u$ is harmonic function.

Since u is given harmonic conjugate

$$v = \int \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) + c$$

$\frac{\partial u}{\partial x}$ is the term not involving 'x'.

$$v = \int (-6xy dx + (2 + 3y^2) dy) + c$$

$$= -6y \frac{x^2}{2} + 2y + \frac{3y^3}{3} + c$$

$$v = -3yx^2 + 2y + y^3 + c.$$



2) Prove that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic and determine its conjugate harmonic.

sol: $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

Condition is $U_{xx} + U_{yy} = 0$

$$U_x = 3x^2 - 3y^2 + 6x$$

$$U_y = -6xy - 6y$$

$$U_{xx} = 6x + 6$$

$$U_{yy} = -6x - 6$$

$$U_{xx} + U_{yy} = 6x + 6 - 6x - 6$$

$$U_{xx} + U_{yy} = 0$$

Therefore u is a harmonic function.

Since u is given harmonic conjugate v is

$$v = \int \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) + c$$

Here $\frac{\partial u}{\partial x}$ is the term not involving x .

$$v = \int (-(-6xy) - 6y) dx + (-3y^2) dy + c$$

$$= \int 6y \frac{x^2}{2} + 6xy - \frac{3y^3}{3} + c$$

$$v = 3x^2y + 6xy - y^3 + c.$$