3) Determine the region 'D' of the w-plane into which the triangular region D enclosed by the lines $x=0, y=0$, $x+y=1$ is transformed under the transformation $w=2z$.

Sol:- Let $w = u+iv$

$z = x+iy$

Given, $w = 2z \Rightarrow u+iv = 2(x+iy)$

$u+iv = 2x+i2y$

$u = 2x, \quad v = 2y$

Given region (D) whose boundary lines are

$x = 0$

$y = 0$

$x+y = 1$

Transformed region D whose boundary lines are

$u = 0$

$v = 0$

$\frac{u}{2} + \frac{v}{2} = 1$

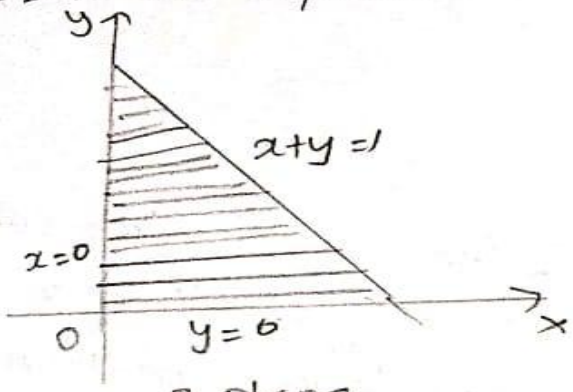
i.e., $u+v = 2$

$\left(x = \frac{u}{2}, \; y = \frac{v}{2}\right)$

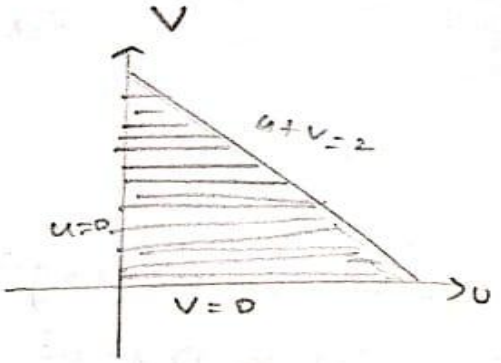
In the z-plane the line $x=0$ is transformed into $u=0$ in the w plane.

In the z-plane the line $y=0$ is transformed into $v=0$ in the w plane.

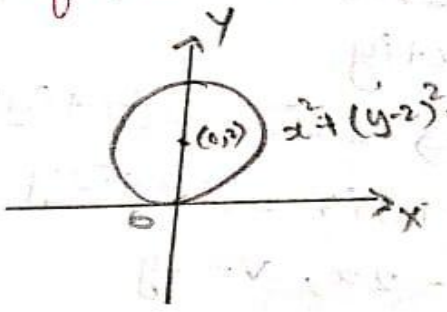
In the z-plane the line $x+y=1$ is transformed into $u+v=2$ in the w plane.



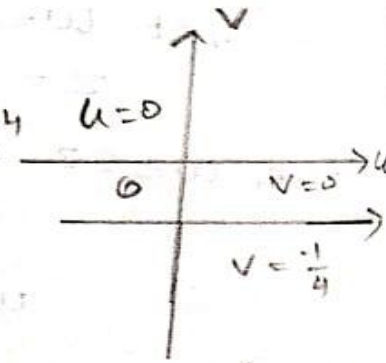
z-plane



4) Find the image of $|z-2i|=2$ under the transformation $w=\frac{1}{z}$.



z-plane



w-plane

Sol. Given $w=\frac{1}{z}$

(i.e.,) $z=\frac{1}{w}$

$w=u+iv$

$$z=\frac{1}{w}=\frac{1}{u+iv}=\frac{u-iv}{(u+iv)(u-iv)}=\frac{u-iv}{u^2+v^2}$$

$$x+iy=\frac{u-iv}{u^2+v^2}$$

$$x=\frac{u}{u^2+v^2},\quad y=\frac{-v}{u^2+v^2}$$

→(1) →(2)

Given: $|z-2i| = 2$

$|x+iy-2i| = 2$

$|x+(y-2)i| = 2$

$x^2+(y-2)^2 = 4$

$x^2+y^2+4-4y = 4$

$x^2+y^2-4y = 0 \rightarrow (3)$

Sub (1), (2) & (3) in equ (3), we get

$$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 - 4\left(\frac{-v}{u^2+v^2}\right) = 0$$

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + \frac{4v}{u^2+v^2} = 0$$

$$\frac{u^2+v^2+4v(u^2+v^2)}{(u^2+v^2)^2} = 0$$

$$\frac{(u^2+v^2)(1+4v)}{(u^2+v^2)^2} = 0$$

$(u^2+v^2 \neq 0)$

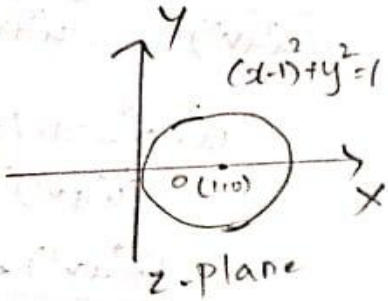
$1+4v = 0$

which is a straight line in w plane.

5) Find the image of the circle $|z-1| = 1$ in the complex Plane under the mapping $w = \frac{1}{z}$.

Sol:- The given transformation $w = \frac{1}{z}$

ie.,) $z = \frac{1}{w}$

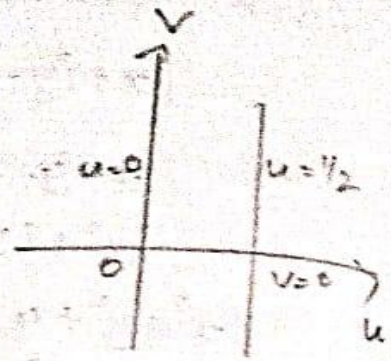


Let $w = u+iv$

$$z = \frac{1}{w} = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$

$$x+iy = \frac{u-iv}{u^2+v^2}$$

$$= \left(\frac{u}{u^2+v^2}\right) + i\left(\frac{-v}{u^2+v^2}\right)$$



Equating the real and imaginary parts,

$$x = \frac{u}{u^2+v^2} \rightarrow (1) \qquad y = \frac{-v}{u^2+v^2} \rightarrow (2)$$

Given: $|z-1| = 1$

$$|x+iy-1| = 1$$

$$|(x-1)+iy| = 1$$

$$(x-1)^2+y^2 = 1$$

$$x^2-2x+1+y^2 = 1$$

$$x^2-2x+y^2 = 0 \rightarrow (3)$$

Sub (1) & (2) in (3) equ, we get

$$\left(\frac{u}{u^2+v^2}\right)^2 - 2\left(\frac{u}{u^2+v^2}\right) + \left(\frac{-v}{u^2+v^2}\right)^2 = 0$$

$$\frac{u^2}{(u^2+v^2)^2} - \frac{2u}{u^2+v^2} + \frac{v^2}{(u^2+v^2)^2} = 0 \qquad u^2+v^2 \neq 0$$

$$\frac{u^2+v^2-2u(u^2+v^2)}{(u^2+v^2)^2} = 0$$

$$\frac{(u^2+v^2)(1-2u)}{(u^2+v^2)^2} = 0$$

$$\frac{1-2u}{u^2+v^2} = 0$$

$$1-2u = 0$$

$$1 = 2u$$

$$\frac{1}{2} = u$$

which is a straight line in w-plane.

6) Find the image of the infinite strips

i) $\frac{1}{4} < y < \frac{1}{2}$ ii) $0 < y < \frac{1}{2}$ under the transformation

$w = \frac{1}{z}$.

Sol:

Let $w = \frac{1}{z}$

$$z = \frac{1}{w} = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$x+iy = \frac{u-iv}{u^2+v^2}$$

$$= \left(\frac{u}{u^2+v^2}\right) + i\left(\frac{-v}{u^2+v^2}\right)$$

$$x = \frac{u}{u^2+v^2} \rightarrow (1) \qquad y = \frac{-v}{u^2+v^2} \rightarrow (2)$$

i) Given strip is $\frac{1}{4} < y < \frac{1}{2}$

when $y = \frac{1}{4} \Rightarrow \frac{1}{4} = \frac{-v}{u^2+v^2}$ [by equ (2)]

$$u^2+v^2 = -4v$$

$$u^2+v^2+4v = 0$$

$$u^2+(v+2)^2-4 = 0$$

$$u^2+(v+2)^2 = 4 \rightarrow (3)$$

which is a circle whose centre is at $(0,-2)$ in the w plane and the radius is 2.

when $y = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{-v}{u^2+v^2}$ [by equ (2)]

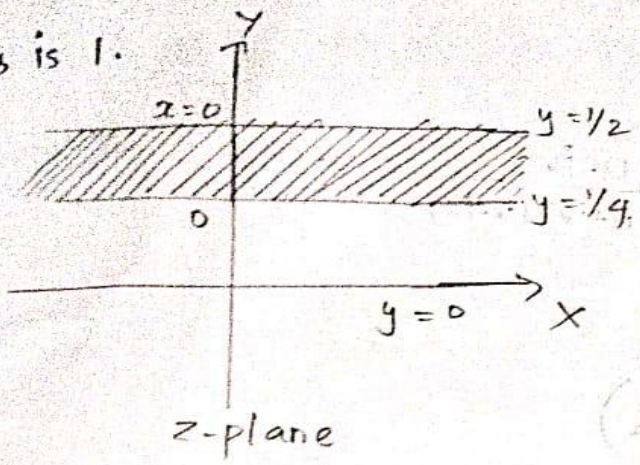
$$u^2+v^2 = -2v$$

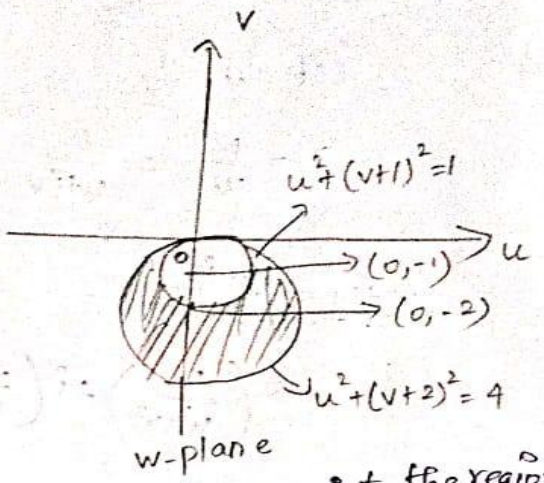
$$u^2+v^2+2v = 0$$

$u^2+(v+1)^2-1=0$

$u^2+(v+1)^2=1 \longrightarrow (4)$

which is a circle whose centre is at $(0,-1)$ in the w plane are radius is 1.





Hence the infinite strip $\frac{1}{4}<y<\frac{1}{2}$ is transformed into the region in between the circles.

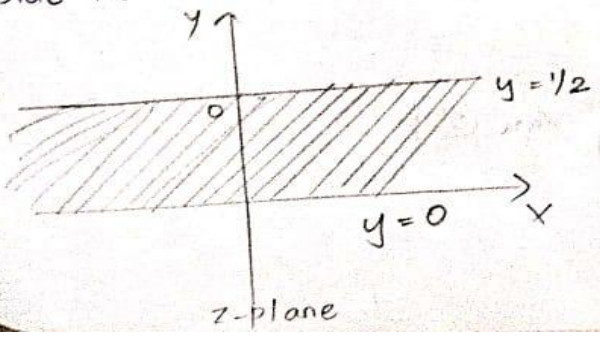
$u^2+(v+1)^2=1$ and $u^2+(v+2)^2=4$ in the w-plane.

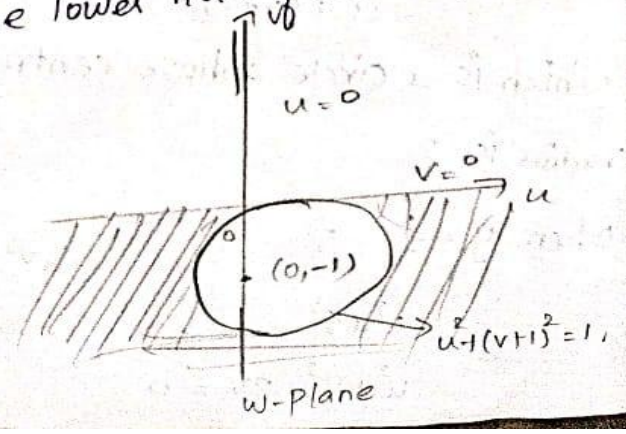
i) Given strip is $0<y<\frac{1}{2}$.

when $y=0 \Rightarrow v=0$ by equ (2).

when $y=\frac{1}{2} \Rightarrow u^2+(v+1)^2=1$ by equ (4)

Hence the infinite strip is $0<y<\frac{1}{2}$ is mapped into the region outside the circle $u^2+(v+1)^2=1$ in the lower half of the w-plane.





7) Determine the image of $1 < x < 2$ under the mapping $w = \frac{1}{z}$.

Sol:- Given $w = \frac{1}{z}$

$$z = \frac{1}{w}$$

$$z = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$x+iy = \frac{u}{u^2+v^2} + i\left(\frac{-v}{u^2+v^2}\right)$$

$$x = \frac{u}{u^2+v^2} \rightarrow (1) \qquad y = \frac{-v}{u^2+v^2} \rightarrow (2)$$

Given :- $1 < x < 2$

When $x = 1 \Rightarrow 1 = \frac{u}{u^2+v^2}$ [by equ (1)]

$$u^2+v^2 = u$$

$$u^2+v^2-u = 0$$

$$\left(u-\frac{1}{2}\right)^2+v^2-\frac{1}{4} = 0$$

$$\left(u-\frac{1}{2}\right)^2+v^2 = \left(\frac{1}{2}\right)^2 \rightarrow (3)$$

which is a circle whose centre is $\left(\frac{1}{2}, 0\right)$ and radius is $\frac{1}{2}$.

When $x = 2 \Rightarrow 2 = \frac{u}{u^2+v^2}$ (by equ (1))

$$2(u^2+v^2) = u$$

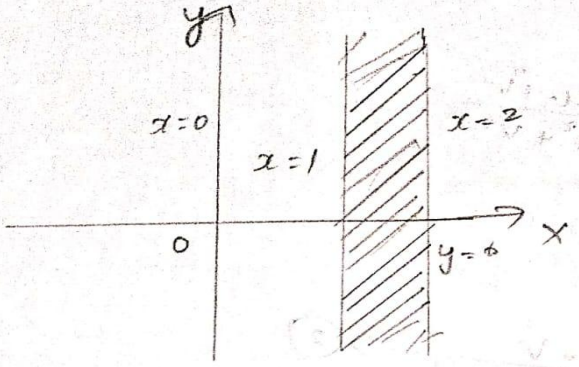
$$u^2+v^2-\frac{u}{2} = 0$$

$$\left(u-\frac{1}{4}\right)^2+v^2-\left(\frac{1}{4}\right)^2 = 0$$

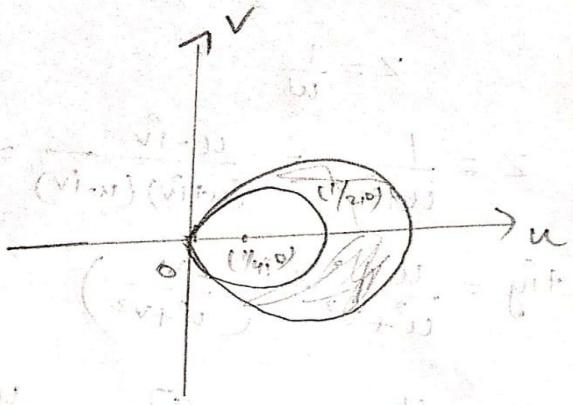
$$\left(u-\frac{1}{4}\right)^2+v^2 = \left(\frac{1}{4}\right)^2 \rightarrow (4)$$

which is a circle whose centre is $(\frac{1}{4}, 0)$ and radius is $\frac{1}{4}$.

Hence the infinite strip $1 < x < 2$ is transformed into the region inbetween the circles (3) to (4) in the w plane.



Z-plane



w plane