



SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Approved by AICTE, New Delhi, Affiliated to Anna University, Chennai

Accredited by NAAC-UGC with 'A++' Grade (Cycle III) &

Accredited by NBA (B.E - CSE, EEE, ECE, Mech & B.Tech.IT)





Bilinear Transformation:-

The transformation $w = \frac{az+b}{cz+d}$, $ad-bc \neq 0$

Where a, b, c, d are complex numbers is called a bilinear transformation.

It is also called as Mobius transformation (or) linear fractional transformation.

Fixed Points (or) Invariant Points:

The fixed points of the transformation $w = \frac{az+b}{cz+d}$

is obtained from $z = \frac{az+b}{cz+d}$ (or) $cz^2 + (d-a)z - b = 0$.

Note:

1. The bilinear transformation which transforms z_1, z_2, z_3 into w_1, w_2, w_3 is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

2. The Cross ratio of four points

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

Problems:-

1) Find the fixed points of $w = \frac{2z+5}{z-4i}$.

sol: The fixed points are $z = \frac{2z+5}{z-4i}$

$$z^2 - 4zi = 2z + 5$$



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$$z^2 - 6zi - 5 = 0$$

$$z = \frac{6i \pm \sqrt{-36 + 20}}{2}$$

$$z = 5i, i$$

2) Find the invariant points of the bilinear transformation

$$\frac{z-1}{z+1}$$

Sol: The invariant points are given by $z = \frac{z-1}{z+1}$

$$z^2 + z = z - 1$$

$$z^2 = 1$$

$$z = \pm \sqrt{1}$$

$$= i, -i$$

3) Obtain the invariant points of the transformation $w = 2 - \frac{2}{z}$

Sol: The invariant points are given by

$$z = 2 - \frac{2}{z}, \quad z = \frac{2z-2}{z}$$

$$z^2 = 2z - 2, \quad z^2 - 2z + 2 = 0$$

$$z = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

4) Find the fixed points:

$$i) w = \frac{z-1-i}{z+2}$$

$$\text{Ans: } z = \frac{-1 \pm i\sqrt{3+4i}}{2}$$

$$ii) w = \frac{2z-5}{z+4}$$

$$\text{Ans: } -1 \pm 2i$$

$$iii) w = \frac{z-2}{z+3}$$

$$\text{Ans: } z = -1 \pm i$$

$$iv) w = \frac{1}{z-2i} \quad \text{Ans: } z = i$$

$$v) w = \frac{5z+4}{z+5} \quad \text{Ans: } z = \pm 2$$



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Problems:

1) Find the bilinear transformation that maps the points $\infty, i, 0$ onto $0, i, \infty$ respectively.

Sol: Given: $z_1 = \infty, z_2 = i, z_3 = 0$ and $w_1 = 0, w_2 = i, w_3 = \infty$.
Let the transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)w_3\left(\frac{w_2}{w_3}-1\right)}{w_3\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{z_1\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)z_1\left(\frac{z_2}{z_1}-1\right)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{\left(\frac{z_1}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\left(\frac{z_2}{z_1}-1\right)}$$

$$\frac{(w-0)(0-1)}{(0-1)(i-0)} = \frac{(0-1)(i-0)}{(z-0)(0-1)}$$

$$\frac{w}{i} = \frac{i}{z}$$

$$wz = i^2$$

$$wz = -1$$

$$w = \frac{-1}{z}$$



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2) Determine the bilinear transformation that maps the points $-1, 0, i$ in the z -plane onto the points $0, i, 3i$ in a w -plane.

Sol:- Given $z_1 = -1, z_2 = 0, z_3 = i$

$w_1 = 0, w_2 = i, w_3 = 3i$

Let the transformation be $\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$

$$\frac{(w-0)(i-3i)}{(w-3i)(i-0)} = \frac{(z-(-1))(0-i)}{(z-i)(0-(-1))}$$

$$\frac{w(-2i)}{(w-3i)(i)} = \frac{(z+1)(-i)}{(z-i)(1)}$$

$$\frac{-2w}{w-3i} = -\frac{z+1}{z-i}$$

$$\frac{2w}{w-3i} = \frac{z+1}{z-i}$$

$$2wz - 2w = wz + w - 3iz - 3i$$

$$2wz - 2w - wz - w = -3i(z+1)$$

$$w(2z - 2 - z - 1) = -3i(z+1)$$

$$w(z-3) = -3i(z+1)$$

$$w = -3i \frac{(z+1)}{(z-3)}$$

3) Find the bilinear transformation which maps the points $-2, 0, 2$ into the points $w = 0, i, -i$ respectively.

Sol:- Given: $z_1 = -2, z_2 = 0, z_3 = 2$

$w_1 = 0, w_2 = i, w_3 = -i$



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Let the transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(i-(-i))}{(w-(-i))(i-0)} = \frac{(z-(-2))(0-2)}{(z-2)(0-(-2))}$$

$$\frac{w(2i)}{(w+i)(i)} = \frac{(z+2)(-2)}{(z-2)(2)}$$

$$\frac{2wi}{(w+i)(i)} = \frac{-2z-4}{2z-4}$$

$$\frac{2w}{w+i} = -\frac{(z+2)}{(z-2)}$$

$$2wz-4w = -zw-z^2-2w-2i^2$$

$$2wz-4w+zw+2w = -z^2-2i^2$$

$$w(2z-4+z+2) = -i(z+2)$$

$$w(3z-2) = -i(z+2)$$

$$w = \frac{-i(z+2)}{3z-2}$$

4) Find the bilinear transformation which maps the points $1, i, -1$ of the z -plane onto the points $0, 1, \infty$ of the w -plane. Show that the transformation maps the interior of the unit circle of the z -plane onto the upper half of the w -plane.

Sol: Given $z_1 = 1, z_2 = i, z_3 = -1$

$w_1 = 0, w_2 = 1, w_3 = \infty$



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Let the transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)w_3}{w_3\left(\frac{w_2}{w_3}-1\right)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(0-1)}{(0-1)(1-0)} = \frac{(z-1)(1+1)}{(z+1)(1-1)}$$

$$w = \frac{(z-1)(1+1)}{(z+1)(1-1)}$$

$$w = i \frac{(z-1)}{(z+1)}$$

Thus the region $|z| < 1$ gives onto the region $\left|\frac{w-1}{w+1}\right| < 1$.

$$\text{Let } w = u+iv \text{ we get } \left|\frac{u+iv-1}{u+iv+1}\right| < 1$$

$$|u+iv-1| < |u+iv+1|$$

$$|u+i(v-1)| < |u+i(v+1)|$$

$$u^2+(v-1)^2 < u^2+(v+1)^2$$

$$(v-1)^2 < (v+1)^2$$



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$$V^2 - 2V + 1 < V^2 + 2V + 1$$

$$V^2 - 2V + 1 - V^2 - 2V - 1 < 0$$

$$-4V < 0$$

$$\text{ie., } V > 0$$

The interior of unit circle is mapped onto the upper half of the w plane.

$$5) 0, i, \infty \text{ onto } \infty, i, 0$$

$$6) z_1 = 2, z_2 = i, z_3 = -2, \text{ into } w_1 = 1, w_2 = i, w_3 = -1.$$