



Construction of conjugate Harmonic function:

* If the real part u is given, then

$$v = \int \left[-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right]$$

* If the imaginary part v is given, then

$$u = \int \left[\frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy \right]$$

J. Show that $u = y + e^x \cos y$ is harmonic and hence find its conjugate harmonic.

Soln.

Given $u = y + e^x \cos y$

$$\begin{aligned} \frac{\partial u}{\partial x} &= e^x \cos y & \left| \frac{\partial u}{\partial y} &= 1 - e^x \sin y \right. \\ \frac{\partial^2 u}{\partial x^2} &= e^x \cos y & \left| \frac{\partial^2 u}{\partial y^2} &= -e^x \cos y \right. \end{aligned}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y = 0$$

Hence u satisfies Laplace eqn.

$\therefore u$ is harmonic.

Now

$$v = \int \left[-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right]$$

$$= \int \left[-(1 - e^x \sin y) dx + e^x \cos y dy \right]$$

$$= \int -dx + \int e^x \sin y dx + \int e^x \cos y dy$$

$$= -x + e^x \sin y + e^x \sin y + C$$

$$v = 2e^x \sin y - x + C$$



2]. Show that $u = \cos x \cosh y$ is harmonic
Find its conjugate harmonic.

Soln.

Given $u = \cos x \cosh y$

$$\begin{aligned} \frac{\partial u}{\partial x} &= -\sin x \cosh y & \left| \frac{\partial u}{\partial y} &= \cos x \sinh y \right. \\ \frac{\partial^2 u}{\partial x^2} &= -\cos x \cosh y & \left| \frac{\partial^2 u}{\partial y^2} &= \cos x \cosh y \right. \end{aligned}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\cos x \cosh y + \cos x \cosh y = 0$$

Hence u satisfies Laplace eqn.

$\therefore u$ is harmonic.

Now $v = \int \left[-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right]$

$$= \int \left[-\cos x \sinh y dx - \sin x \cosh y dy \right]$$

$$= -\sin x \sinh y + \cos x \sin x \sinh y dx + C$$

$$v = -2 \sin x \sinh y + C$$

3]. Prove that $u = x^3 - 3xy^2 + 3x^2 - 3y^2$ is harmonic function. Find conjugate harmonic function

Soln.

Given $u = x^3 - 3xy^2 + 3x^2 - 3y^2$

$$\begin{aligned} \frac{\partial u}{\partial x} &= 3x^2 - 3y^2 + 6x & \left| \frac{\partial u}{\partial x^2} &= 6x + 6 \right. \\ \frac{\partial u}{\partial y} &= -6xy - 6y & \left| \frac{\partial^2 u}{\partial y^2} &= -6x - 6 \right. \end{aligned}$$



$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + 6 - 6x - 6$$
$$= 0$$

Hence u satisfies Laplace eqn.

$\therefore u$ is harmonic.

Now,

$$v = \int \left[-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right]$$

$$= \int \left[(-6xy - 6y) dx + (3x^2 - 3y^2 + 6x) dy \right]$$

$$= \int (6xy + 6y) dx + (3x^2 - 3y^2 + 6x) dy$$

$$= \frac{6x^2y}{2} + 6xy + 3x^2y - \frac{3y^3}{3} + 6xy$$

$$v = 6x^2y + 12xy - y^3 + C$$