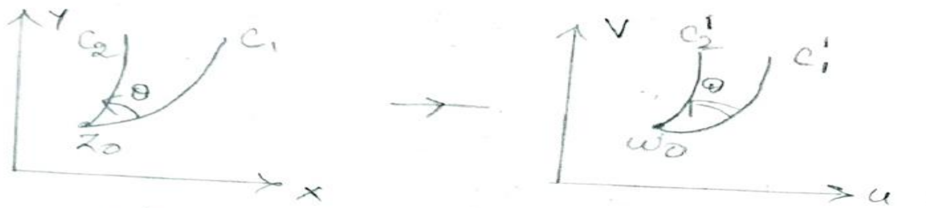
Conformal mapping:

A transformation that preserves angles between every pair of curves through a point both in magnitude and sense, is said to be conformal at that point.



z-plane w-plane

Isogonal mapping:

A transformation under which angles between every pair of curves through a point are preserved in magnitude, but different in direction.

z-plane w-plane

Critical point:

The point z_0 is said to be critical point of the transformation $w = f(z)$ if $f(z)$ is not conformal at z_0. The critical point is defined by,

$$\frac{dw}{dz} = 0 \quad \text{and} \quad \frac{dz}{dw} = 0$$

1]. Find the critical point of the transformation $w = z + \frac{1}{z}$

Soln.

Given $w = z + \frac{1}{z}$

$$\frac{dw}{dz} = 1 - \frac{1}{z^2} = \frac{z^2-1}{z^2}$$

$$\frac{dz}{dw} = \frac{z^2}{z^2-1}$$

Now $\frac{dw}{dz} = 0 \Rightarrow \frac{z^2-1}{z^2} = 0$

$$z^2 - 1 = 0$$

$$z^2 = 1$$

$$z = \pm 1$$

and $\frac{dz}{dw} = 0 \Rightarrow \frac{z^2}{z^2-1} = 0 \Rightarrow z^2 = 0$

$$z = 0$$

The critical points are $0, \pm 1$.

2]. Find the critical point of $w^2 = (z-\alpha)(z-\beta)$

Soln.

Given $w^2 = (z-\alpha)(z-\beta)$

Differentiate w.r. to z,

$$2w\frac{dw}{dz} = (z-\alpha)(1) + (z-\beta)(1)$$

$$= (z-\alpha) + (z-\beta)$$

$$\frac{dw}{dz} = \frac{2z-(\alpha+\beta)}{2w}$$

and $\frac{dz}{dw} = \frac{2w}{2z-(\alpha+\beta)}$

Now $\frac{dw}{dz} = 0 \Rightarrow \frac{2z-(\alpha+\beta)}{2w} = 0$

$$2z = \alpha + \beta$$

$$z = \frac{\alpha+\beta}{2}$$

and $\frac{dz}{dw} = 0 \Rightarrow \frac{2w}{2z-(\alpha+\beta)} = 0$

$$2w = 0$$
$$w = 0$$
$$\sqrt{(z-\alpha)(z-\beta)} = 0$$
$$(z-\alpha)(z-\beta) = 0$$
$$z = \alpha \text{ and } z = \beta$$

$\therefore$ The critical points are $z = \frac{\alpha+\beta}{2}, \alpha, \beta$.

3]. Find the critical point of $w = z^4 - 1$

Soln.

Given $w = z^4 - 1$

Now, $\frac{dw}{dz} = 4z^3 \Rightarrow \frac{dw}{dz} = 0 \Rightarrow 4z^3 = 0$

$$z = 0$$

and $\frac{dz}{dw} = \frac{1}{4z^3} = 0$, which is not possible

$\therefore$ The critical point is 0.

4]. Find the critical point of $w = \sin z$

Soln.

Given $w = \sin z$

$\frac{dw}{dz} = \cos z \Rightarrow \frac{dw}{dz} = 0$

$$\cos z = 0$$
$$z = \cos^{-1}(0)$$
$$z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$$
$$z = \pm(2n-1)\frac{\pi}{2} \quad \text{where } n = 1, 2, 3, \ldots$$

and $\frac{dz}{dw} = 0 \Rightarrow \frac{1}{\cos z} = 0$, which is not possible.

Hw 1. $w = \cos z$

Fixed point:

A point $z=a$ is said to be fixed point of a mapping $w=f(z)$ if its image under $f(z)$ is itself.

i.e., $f(z)=z$

1] Find the invariant points of $w=\frac{1+z}{1-z}$

Soln.

Given $w=\frac{1+z}{1-z}$

The invariant points are,

$$z=\frac{1+z}{1-z} \quad (\because w=f(z)=z)$$

$$z(1-z)=1+z$$

$$z-z^2=1+z$$

$$z-z^2-1-z=0$$

$$-z^2-1=0$$

$$z^2=-1$$

$$z=\pm i$$

2]. Find the fixed points of $w=\frac{2z+6}{z+7}$

Soln.

Given $w=\frac{2z+6}{z+7}$

The fixed points are,

$$z=\frac{2z+6}{z+7}$$

$$z(z+7)=2z+6$$

$$z^2+7z-2z-6=0$$

$$z^2+5z-6=0$$

$$(z-1)(z+6)=0$$

$$z=1,-6$$