



Mobius Transformation (or) Bilinear Transformation

The transformation $w = \frac{az+b}{cz+d}$, $ad-bc \neq 0$, where a, b, c, d are complex numbers, is called a bilinear transformation.

Formula:

Bilinear transformation of z_1, z_2, z_3 into w_1, w_2, w_3 is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Q. Find the bilinear transformation which maps the points $z=0, -i, -1$ into $w=i, 1, 0$ respectively.

Soln.:

Given $z_1=0, z_2=-i, z_3=-1$

$w_1=i, w_2=1, w_3=0$

The bilinear transformation is,

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$$

$$\frac{(w-i)(1-0)}{(w-0)(1-i)} = \frac{(z-0)(-i+1)}{(z+1)(-i-0)}$$

$$\frac{w-i}{w-wi} = \frac{z(1-i)}{(-i)(z+1)}$$

$$\frac{w-i}{w-wi} = \frac{z-z_i}{-iz-i}$$

$$(w-i)(-iz-i) = (z-zi)(w-wi)$$

$$-wzi - wi - z - 1 = wz - wzi - wz - wz$$

$$-wzi - wi + wz + wz = z + 1$$

$$wzi - wi = z + 1$$

$$wi(z-1) = z + 1$$



$$\omega = \frac{1}{i} \frac{z+i}{z-1}$$

$$= \frac{i}{i^2} \frac{z+i}{z-1}$$

$$\omega = -i \left(\frac{z+i}{z-1} \right)$$

2]. Find the bilinear transformation which maps $z = 1, i, -1$ to $w = i, 0, -i$.

Soln.

Given

$$z_1 = 1, z_2 = i, z_3 = -1$$

$$w_1 = i, w_2 = 0, w_3 = -i$$

The bilinear transformation is,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-i)(0+i)}{(w+i)(0-i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)} \quad \text{(or)} \quad \frac{i(w-i) + (-i)(w+i)}{i(w-i) - (-i)(w+i)}$$

$$\frac{(w-i)}{-(w+i)} = \frac{z-1}{z+1} \left(\frac{1+i}{-1+i} \right)$$

$$= \left(\frac{z-1}{z+1} \right) \frac{(1+i)(-1-i)}{(-1+i)(-1-i)}$$

$$= \left(\frac{z-1}{z+1} \right) \frac{-1-i-i+1}{(-1)^2 - i^2}$$

$$= \frac{z-1}{z+1} \left(\frac{-2i}{2} \right)$$

$$-\left[\frac{w-i}{w+i} \right] = -\left[\frac{z-1}{z+1} \right] i$$

$$\frac{w-i}{w+i} = \frac{iz-i}{z+1}$$

By componendo & dividendo,

$$\frac{(w-i) + (w+i)}{(w-i) - (w+i)} = \frac{(iz-i) + (z+1)}{(iz-i) - (z+1)}$$

$$\begin{aligned} &= \frac{(z-1)(1+i) + (-i)(z+1)}{(z-1)(-1+i) - (-i)(z+1)} \\ &\downarrow \\ w &= \frac{-iz-1}{iz-1} \end{aligned}$$



$$\Rightarrow \frac{2w}{-2i} = \frac{z(i+1) + (1-i)}{z(i-1) - (1+i)}$$

$$w = -i \frac{z(i+1) + (1-i)}{z(i-1) - (1+i)}$$

$$w = \frac{-2i + z(i+1)}{-1-i}$$

Find the bilinear transformation which maps $-1, -i, 1$ in z -plane to $\infty, i, 0$ in w -plane respectively.

Soln.

Given $z_1 = -1, z_2 = -i, z_3 = 1$

$w_1 = \infty, w_2 = i, w_3 = 0$

The Bilinear transformation is,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$w_1 = \left(\frac{w}{w_1} - 1\right)(w_2 - w_3) = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(0-1)(i-0)}{w(0-1)} = \frac{(z+1)(-i-1)}{(z-1)(-i+1)}$$

$$\frac{-1}{-w} = \frac{(z+1)(-i-1)}{(z-1)(-i+1)}$$

$$w = i \frac{(z-1)(-i+1)}{(z+1)(-i-1)}$$

$$= i \frac{[-iz + z + i - 1]}{(-iz - z - i - 1)}$$

$$= \frac{z + iz - 1 - i}{-iz - z - i - 1}$$

$$= \frac{(z-1) + i(z-1)}{(-z-1) + i(-z-1)}$$



$$= \frac{(z-1)(1+i)}{(-z-1)(1+i)}$$

$$w = \frac{1-z}{1+z}$$

$$w = \frac{1-z}{z+1}$$

4]. Find the bilinear transformation which maps $\infty, i, 0$ to $0, i, \infty$.

Soln.

Given $z_1 = \infty, z_2 = i, z_3 = 0$

$w_1 = 0, w_2 = i, w_3 = \infty$

The bilinear transformation is,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)w_3\left(\frac{w_2}{w_3}-1\right)}{w_3\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{z_1\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)z_1\left(\frac{z_2}{z_1}-1\right)}$$

$$\frac{(w-0)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\left(\frac{z_2}{z_1}-1\right)}$$

$$\frac{(w-0)(0-1)}{(0-1)(i-0)} = \frac{(0-1)(i-0)}{(z-0)(i-1)}$$

$$\frac{-w}{-i} = \frac{-i}{-z}$$

$$w = \frac{1}{z}$$



Soln.

$$\text{Given } z_1 = 0, z_2 = 1, z_3 = \infty$$

$$w_1 = -5, w_2 = -1, w_3 = 3$$

The bilinear transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)z_3 \left(\frac{z_2}{z_3} - 1 \right)}{z_3 \left(\frac{z}{z_3} - 1 \right) (z_2 - z_1)}$$

$$\frac{(w+5)(-1-3)}{(w-3)(-1+5)} = \frac{(z-0)(0-1)}{(w-1)(1-0)}$$

$$\frac{(w+5)(-4)}{(w-3)(4)} = \frac{-z}{-1}$$

$$\frac{-(w+5)}{w-3} = \frac{z}{1}$$

$$-(w+5) = z(w-3)$$

$$-w-5 = wz-3z$$

$$wz+w = 3z-5$$

$$w(z+1) = 3z-5$$

$$w = \frac{3z-5}{z+1}$$