Unit - I

Vector Calculus

Scalar quantity:

A scalar quantity is that which has magnitude and is not related to any direction.

Vector quantity:

A vector quantity is that which has both magnitude and direction.

Scalar Point function:

If corresponding to each point P of a region R there corresponds a scalar denoted by $\phi(P)$ or $\phi(x,y,z)$ then ϕ is said to be a scalar point function for the region R.

Example: The temperature $\phi(P)$ at any point P of a body occupying a certain region is a scalar point function.

Vector Point function:

If corresponding to each point P of a region R, there corresponds a vector denoted by $F(P)$, then F is said to be a vector point function for the region R.

Example: The acceleration $F(P)$ of a particle at any time t occupying the position P in a certain region is a vector point function.

Vector Differential operator:

The vector differential operator ∇ is defined as.

$$\boxed{\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} = \sum \vec{i}\frac{\partial}{\partial x}}$$

Gradient of a Scalar Point function:

Let $\phi(x,y,z)$ be a Scalar Point function and is Continuously differentiable then the Vector,

$$\boxed{\begin{aligned}\nabla\phi &= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi \\ &= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\end{aligned}}$$

is called the gradient of ϕ and is written as grad ϕ.

$$\boxed{\text{ie., grad } \phi = \nabla\phi}$$

Note:

1. $\nabla\phi$ defines a Vector field.
2. $\nabla\phi \neq \phi\nabla$ then there will be no '.' or '$\times$' between ϕ and ∇.

Properties of Gradient:

1. If f and g are two scalar point functions then,

$$\boxed{\begin{aligned}&\nabla(f \pm g) = \nabla f \pm \nabla g \\ &\text{(or) grad } (f \pm g) = \text{grad } f \pm \text{grad } g\end{aligned}}$$

2. If f and g are two Scalar point function then,

$$\boxed{\begin{aligned}&\nabla(fg) = f\nabla g + g\nabla f \\ &\text{(or) grad } (fg) = f(\text{grad } g) + g(\text{grad } f)\end{aligned}}$$

3. If f and g are two scalar Point function then,

$$\boxed{\begin{aligned}&\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2} \quad \text{where } g \neq 0 \\ &\text{(or) grad } \left(\frac{f}{g}\right) = \frac{g(\text{grad } f) - f(\text{grad } g)}{g^2}\end{aligned}}$$

4. Gradient of a constant is zero.

ie.,) $\nabla\phi = 0$

Problems:

1) Find grad ϕ where $\phi = x^2+y^2+z^2$

Solution:

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2+y^2+z^2) + \vec{j}\frac{\partial}{\partial y}(x^2+y^2+z^2) + \vec{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$= \vec{i}(2x) + \vec{j}(2y) + \vec{k}(2z)$$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\boxed{\nabla\phi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}}$$

2) Find grad ϕ if $\phi = xyz$ at $(1,1,1)$

Solution:

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(xyz) + \vec{j}\frac{\partial}{\partial y}(xyz) + \vec{k}\frac{\partial}{\partial z}(xyz)$$

$$= \vec{i}(yz) + \vec{j}(xz) + \vec{k}(xy)$$

$$\nabla\phi \text{ at } (1,1,1) = \vec{i}(1\times1) + \vec{j}(1\times1) + \vec{k}(1\times1)$$

$$= \vec{i}+\vec{j}+\vec{k}$$

$$\boxed{\nabla\phi_{(1,1,1)} = \vec{i}+\vec{j}+\vec{k}}$$

3) Find grad ϕ where $\phi = 3x^2y - y^3z^2$ at $(1,1,1)$.

Solution:

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \vec{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \vec{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= \vec{i}(6xy) + \vec{j}(3x^2 - 3y^2z^2) + \vec{k}(-2y^3z)$$

$$\nabla\phi \text{ at } (1,1,1) = \vec{i}(6\times1\times1) + \vec{j}(3\times1 - 3\times1\times1) + \vec{k}(-2\times1\times1)$$

$$= 6\vec{i} + \vec{j}(3-3) + \vec{k}(-2)$$

$$= 6\vec{i} - 2\vec{k}$$

$$\boxed{\nabla\phi_{(1,1,1)} = 6\vec{i} - 2\vec{k}}$$

4) If $\phi = \log(x^2+y^2+z^2)$ find $\nabla\phi$.

Solution:

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}\left[\log(x^2+y^2+z^2)\right] + \vec{j}\frac{\partial}{\partial y}\left[\log(x^2+y^2+z^2)\right] + \vec{k}\frac{\partial}{\partial z}\left[\log(x^2+y^2+z^2)\right]$$

$$= \vec{i}\frac{1}{x^2+y^2+z^2}(2x) + \vec{j}\frac{1}{x^2+y^2+z^2}(2y) + \vec{k}\frac{1}{x^2+y^2+z^2}(2z)$$

$$= \frac{2}{x^2+y^2+z^2}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\boxed{\nabla\phi = \frac{2\vec{r}}{x^2+y^2+z^2}} \quad \text{where } \boxed{\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}}$$

5) Find $\nabla(\log r)$

Solution:

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\nabla(\log r) = \vec{i}\frac{\partial}{\partial x}(\log r) + \vec{j}\frac{\partial}{\partial y}(\log r) + \vec{k}\frac{\partial}{\partial z}(\log r)$$

$$= \vec{i}\frac{1}{r}\frac{\partial r}{\partial x} + \vec{j}\frac{1}{r}\frac{\partial r}{\partial y} + \vec{k}\frac{1}{r}\frac{\partial r}{\partial z} \rightarrow (1)$$

Since $r = \sqrt{x^2+y^2+z^2}$

$$\frac{\partial r}{\partial x} = \frac{1}{\cancel{2}\sqrt{x^2+y^2+z^2}} \cdot \cancel{2}x = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$$

Similarly, $\boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$ and $\boxed{\frac{\partial r}{\partial z} = \frac{z}{r}}$

Substitute these values in (1),

$$\nabla(\log r) = \vec{i}\left(\frac{1}{r}\right)\left(\frac{x}{r}\right) + \vec{j}\left(\frac{1}{r}\right)\left(\frac{y}{r}\right) + \vec{k}\left(\frac{1}{r}\right)\left(\frac{z}{r}\right)$$

$$= \vec{i}\frac{x}{r^2} + \vec{j}\frac{y}{r^2} + \vec{k}\frac{z}{r^2}$$

$$= \frac{1}{r^2}\left[x\vec{i} + y\vec{j} + z\vec{k}\right]$$

$$= \frac{\vec{r}}{r^2}$$

$$\boxed{\nabla(\log r) = \frac{\vec{r}}{r^2}}$$

$\phi = x^2y^2z^2 + 4xz^2 + 7y \quad (1,2,7)$

6) If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ such that $|\vec{r}| = r$ Prove that

i) $\nabla r = \frac{\vec{r}}{r} = \hat{r}$

ii) $\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3} = -\frac{\hat{r}}{r^2}$

iii) $\nabla r^n = n r^{n-2}\vec{r}$

iv) $\nabla f(r) = f'(r)\nabla r$

v) $\nabla f(r) \times \vec{r} = 0$

vi) If $\nabla\phi$ is Solenoidal find $\nabla^2\phi$.

Solution:

i) Given: $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$$

$$\boxed{r^2 = x^2+y^2+z^2}$$

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$$

$$2r\frac{\partial r}{\partial y} = 2y \Rightarrow \boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$$

$$2r\frac{\partial r}{\partial z} = 2z \Rightarrow \boxed{\frac{\partial r}{\partial z} = \frac{z}{r}}$$

$$\nabla r = \vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}$$

$$= \vec{i}\left(\frac{x}{r}\right) + \vec{j}\left(\frac{y}{r}\right) + \vec{k}\left(\frac{z}{r}\right)$$

$$= \frac{1}{r}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) \qquad \boxed{\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}}$$

$$\boxed{\nabla r = \frac{\vec{r}}{r} = \hat{r}}$$

ii) $\nabla\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$

$$\nabla\left(\frac{1}{r}\right) = \vec{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \vec{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \vec{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= \vec{i}\left(\frac{-1}{r^2}\frac{\partial r}{\partial x}\right) + \vec{j}\left(\frac{-1}{r^2}\frac{\partial r}{\partial y}\right) + \vec{k}\left(\frac{-1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= \frac{-1}{r^2}\left[\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right]$$

$$\nabla\left(\frac{1}{r}\right) = \frac{-1}{r^3}\left(\vec{r}\right)$$

$$= \frac{-1}{r^2}\left(\frac{\vec{r}}{r}\right)$$

$$= -\frac{\hat{r}}{r^2}$$

$$\boxed{\nabla\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3} = -\frac{\hat{r}}{r^2}}$$

iii) $\nabla r^n = n r^{n-2}\vec{r}$

$$\nabla r^n = \left(\vec{i}\frac{\partial r^n}{\partial x} + \vec{j}\frac{\partial r^n}{\partial y} + \vec{k}\frac{\partial r^n}{\partial z}\right)$$

$$= \vec{i}\, n r^{n-1}\frac{\partial r}{\partial x} + \vec{j}\, n r^{n-1}\frac{\partial r}{\partial y} + \vec{k}\, n r^{n-1}\frac{\partial r}{\partial z}$$

$$= n r^{n-1}\left[\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right]$$

$$= \frac{n r^{n-1}}{r}\left[x\vec{i} + y\vec{j} + z\vec{k}\right]$$

$$= n r^{n-2}\vec{r}$$

$$\boxed{\nabla r^n = n r^{n-2}\vec{r}}$$

~~iv) $\nabla f(r) \times \vec{r} = 0$~~

iv) $\nabla f(r) = f'(r)\nabla r$

$$\nabla f(r) = \vec{i}\frac{\partial}{\partial x} f(r) + \vec{j}\frac{\partial}{\partial y} f(r) + \vec{k}\frac{\partial}{\partial z} f(r)$$

$$= \vec{i} f'(r)\frac{\partial r}{\partial x} + \vec{j} f'(r)\frac{\partial r}{\partial y} + \vec{k} f'(r)\frac{\partial r}{\partial z}$$

$$= f'(r)\left[\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right]$$

$$= \frac{f'(r)}{r}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$= \frac{f'(r)}{r}\vec{r}$$

$$\boxed{\nabla f(r) = \frac{f'(r)}{r}\vec{r}}$$

v) $\nabla f(r) \times \vec{r} = 0$

$$\nabla f(r) \times \vec{r} = \frac{f'(r)}{r}\vec{r} \times \vec{r}$$

$$= \frac{1}{r} f'(r)\left[\vec{r} \times \vec{r}\right]$$

$$\boxed{(\because \vec{r} \times \vec{r} = 0)}$$

$$= 0$$

$$\boxed{\nabla f(r) \times \vec{r} = 0}$$

vi) $\nabla^2 \phi = 0$

$$\nabla^2 \phi = \nabla(\nabla\phi) \quad (\because \nabla\phi \text{ is Solenoidal } \nabla\cdot\phi = 0)$$

$$= \nabla(0)$$

$$= 0 \quad \boxed{\nabla^2\phi = 0}$$

Level Surface: Important Results:

Unit Normal:

A Unit normal to the given surface ϕ at the Point is $\frac{\nabla\phi}{|\nabla\phi|}$.

Directional Derivative:

The directional derivative of ϕ in the direction $\vec{a}$ is given by,

$$\nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} \quad (\text{or}) \quad \nabla\phi \cdot \hat{n} \quad \text{where } \hat{n} = \frac{\vec{a}}{|\vec{a}|}$$

The directional derivative is maximum in the direction of the normal to the given surface. Its maximum value is $|\nabla\phi|$.

Angle between two surfaces:

$$\boxed{\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1| \cdot |\nabla\phi_2|}}$$

Note:

If the surfaces cut orthogonally then

$$\boxed{\nabla\phi_1 \cdot \nabla\phi_2 = 0}$$

Problems:

1) Find a unit normal to the surface $x^2y + 2xz = 4$ at $(2,-2,3)$

Solution:

$$\phi : x^2y + 2xz - 4$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2y+2xz-4)+\vec{j}\frac{\partial}{\partial y}(x^2y+2xz-4)+\vec{k}\frac{\partial}{\partial z}(x^2y+2xz-4)$$

$$= \vec{i}(2xy+2z)+\vec{j}(x^2)+\vec{k}(2x)$$

$$\nabla\phi_{(2,-2,3)} = \vec{i}(-8+6)+\vec{j}(4)+\vec{k}(4)$$

$$= -2\vec{i}+4\vec{j}+4\vec{k}$$

$$|\nabla\phi| = \sqrt{4+16+16} = \sqrt{36} = 6$$

Unit normal to the given Surface at $(2,-2,3)$

$$= \frac{\nabla\phi}{|\nabla\phi|} = \frac{-2\vec{i}+4\vec{j}+4\vec{k}}{6}$$

$$= \frac{1}{3}(-\vec{i}+2\vec{j}+2\vec{k})$$

$$\boxed{\text{Unit normal} = \frac{1}{3}(-\vec{i}+2\vec{j}+2\vec{k})}$$

2) Find the unit vector normal to $x^2-y^2+z=2$ at $(1,-1,2)$

Sol :- $\dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{2\vec{i}+2\vec{j}+\vec{k}}{3}$

3) Find the unit vector normal to $x^2+xy+z^2=4$ at $(1,-1,2)$

Sol :- $\dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{\vec{i}+\vec{j}+4\vec{k}}{\sqrt{18}}$

4) Find the directional derivative of the function x^2+2xy at $(1,-1,3)$ in the direction $\vec{i}+2\vec{j}+2\vec{k}$.

Sol:- $\phi = x^2+2xy$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2+2xy)+\vec{j}\frac{\partial}{\partial y}(x^2+2xy)+\vec{k}\frac{\partial}{\partial z}(x^2+2xy)$$

$$= \vec{i}(2x+2y)+\vec{j}(2x)+\vec{k}(0)$$

$$= \vec{i}(2x+2y)+\vec{j}(2x)$$

$$\nabla\phi_{(1,-1,3)} = \vec{i}(2(1)+2(-1))+\vec{j}(2(1))$$

$$= \vec{i}(2-2)+\vec{j}(2)$$

$$\boxed{\nabla\phi_{(1,-1,3)} = 2\vec{j}}$$

Given: $\vec{a} = \vec{i}+2\vec{j}+2\vec{k}$

$$|\vec{a}| = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$\hat{n} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{i}+2\vec{j}+2\vec{k}}{3}$$

$$\nabla\phi.\hat{n} = 2\vec{j}\cdot\left[\frac{\vec{i}+2\vec{j}+2\vec{k}}{3}\right]$$

$$= \frac{4}{3}$$

$$\boxed{\nabla\phi.\hat{n} = \frac{4}{3}}$$

5) Find the directional derivative of $xy+yz+zx$ at $(1,1,1)$ in the direction $\vec{i}+\vec{j}$.

Sol: $2\sqrt{2}$

6) Find the directional derivative of $3x^2+2y-3z$ at $(1,1,1)$ in the direction $2\vec{i}+2\vec{j}-\vec{k}$.

Sol: $\frac{19}{3}$

7) What is the greatest rate of increase of $\phi = xyz^2$ at $(1,0,3)$

Sol: Let $\phi = xyz^2$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(xyz^2) + \vec{j}\frac{\partial}{\partial y}(xyz^2) + \vec{k}\frac{\partial}{\partial z}(xyz^2)$$

$$= \vec{i}(yz^2) + \vec{j}(xz^2) + \vec{k}(2xyz)$$

$$\nabla\phi_{(1,0,3)} = \vec{i}(0) + \vec{j}(9) + \vec{k}(0)$$

$$= 9\vec{j}$$

$$\boxed{\nabla\phi_{(1,0,3)} = 9\vec{j}}$$

Maximum (or) Greatest rate of increase $= |\nabla\phi|$

$$= \sqrt{9^2}$$

$$= \sqrt{81}$$

$$= 9$$

$$\boxed{\text{Greatest rate of increase} = 9}$$

8) In what direction from the point $(1,-1,2)$ is the directional derivative of $\phi = x^2y^2z^3$ a maximum? What is the magnitude of this maximum?

Sol:

$$\phi = x^2y^2z^3$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2y^2z^2) + \vec{j}\frac{\partial}{\partial y}(x^2y^2z^2) + \vec{k}\frac{\partial}{\partial z}(x^2y^2z^2)$$

$$= 2xy^2z^2\vec{i} + 2x^2yz^2\vec{j} + 2x^2y^2z\vec{k}$$

$$\boxed{\nabla\phi_{(1,-1,2)} = 16\vec{i} - 16\vec{j} + 12\vec{k}}$$ is the directional derivative.

Magnitude is $|\nabla\phi| = \sqrt{16^2+16^2+12^2}$

$$= \sqrt{256+256+144}$$

$$= \sqrt{656}$$

$$\boxed{\text{Magnitude} = |\nabla\phi| = \sqrt{656}}$$

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9) Find the directional derivative of $\phi = xy^2z^3$ at the point $(1,1,1)$ along the normal to the surface $x^2+xy+z^2=3$ at the point $(1,1,1)$.

Sol: $\nabla\phi$ is normal to the surface x^2+xy+z^2-3

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2+xy+z^2-3) + \vec{j}\frac{\partial}{\partial y}(x^2+xy+z^2-3) + \vec{k}\frac{\partial}{\partial z}(x^2+xy+z^2-3)$$

$$= \vec{i}(2x+y) + \vec{j}(x) + \vec{k}(2z)$$

$$\boxed{\nabla\phi_{(1,1,1)} = 3\vec{i} + \vec{j} + 2\vec{k}}$$

To find the directional derivative of $\phi = xy^2z^3$ at (1,1,1) in the direction $\vec{a} = 3\vec{i}+\vec{j}+2\vec{k}$.

$$\nabla\phi = \vec{i}\frac{\partial}{\partial x}(xy^2z^3) + \vec{j}\frac{\partial}{\partial y}(xy^2z^3) + \vec{k}\frac{\partial}{\partial z}(xy^2z^3)$$

$$= \vec{i}(y^2z^3) + \vec{j}(2xyz^3) + \vec{k}(3xy^2z^2)$$

$$\nabla\phi_{(1,1,1)} = \vec{i}(1) + \vec{j}(2) + \vec{k}(3)$$

$$\boxed{\nabla\phi_{(1,1,1)} = \vec{i}+2\vec{j}+3\vec{k}}$$

$$\text{Directional derivative} = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= (\vec{i}+2\vec{j}+3\vec{k}) \cdot \frac{(3\vec{i}+\vec{j}+2\vec{k})}{\sqrt{9+1+4}}$$

$$= \frac{3+2+6}{\sqrt{14}}$$

$$= \frac{11}{\sqrt{14}}$$

$$\boxed{\text{Directional derivative} = \frac{11}{\sqrt{14}}}$$

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10) Find the angle between the surfaces $x^2+y^2+z^2=5$ and $x^2+y^2+z^2-2x=5$ at $(0,1,2)$.

<u>Sol:</u> Let $\phi_1 = x^2+y^2+z^2-5$

$$\frac{\partial\phi_1}{\partial x} = 2x$$

$$\frac{\partial\phi_1}{\partial y} = 2y$$

$$\frac{\partial\phi_1}{\partial z} = 2z$$

$\phi_2 = x^2+y^2+z^2-2x-5$

$$\frac{\partial\phi_2}{\partial x} = 2x-2$$

$$\frac{\partial\phi_2}{\partial y} = 2y$$

$$\frac{\partial\phi_2}{\partial z} = 2z$$

$\nabla\phi_1 = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

$\nabla\phi_2 = (2x-2)\vec{i} + 2y\vec{j} + 2z\vec{k}$

$= 2x\vec{i} - 2\vec{i} + 2y\vec{j} + 2z\vec{k}$

$\nabla\phi_1(0,1,2) = 2(0)\vec{i} + 2(1)\vec{j} + 2(2)\vec{k}$

$= 0 + 2\vec{j} + 4\vec{k}$

$\boxed{\nabla\phi_1(0,1,2) = 2\vec{j} + 4\vec{k}}$

$|\nabla\phi_1| = \sqrt{2^2+4^2} = \sqrt{4+16} = \sqrt{20}$

$\boxed{|\nabla\phi_1| = \sqrt{20}}$

$\boxed{\nabla\phi_2(0,1,2) = -2\vec{i} + 2\vec{j} + 4\vec{k}}$

$\nabla\phi_2(0,1,2) = -2\vec{i} + 2\vec{j} + 4\vec{k}$

$\cancel{|\nabla\phi_2| = \sqrt{2^2+4^2+2^2}}$

$|\nabla\phi_2| = \sqrt{2^2+2^2+4^2} = \sqrt{4+4+16} = \sqrt{24} = \sqrt{24}$

$\boxed{|\nabla\phi_2| = \sqrt{24}}$

Angle between the surfaces,

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$$= \frac{(2\vec{j} + 4\vec{k}) \cdot (-2\vec{i} + 2\vec{j} + 4\vec{k})}{\sqrt{20} \cdot \sqrt{24}}$$

$$= \frac{4+16}{\sqrt{20} \cdot \sqrt{24}}$$

$$\cos\theta = \frac{4+16}{\sqrt{20}\cdot\sqrt{24}}$$

$$= \frac{20}{\sqrt{20}\cdot\sqrt{24}}$$

$$= \frac{\sqrt{20}\cdot\cancel{\sqrt{20}}}{\cancel{\sqrt{20}}\cdot\sqrt{24}}$$

$$= \sqrt{\frac{20}{24}}$$

$$\cos\theta = \sqrt{\frac{5}{6}}$$

$$\boxed{\theta = \cos^{-1}\sqrt{\frac{5}{6}}}$$

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11) Find the angle between the surfaces $x\log z = y^2 - 1$ and $x^2 y = 2 - z$ at the Point $(1,1,1)$.

Sol: $\phi_1 = x\log z - y^2 + 1$

$$\frac{\partial\phi_1}{\partial x} = \log z$$

$$\frac{\partial\phi_1}{\partial y} = -2y$$

$$\frac{\partial\phi_1}{\partial z} = \frac{x}{z}$$

$$\nabla\phi_1 = \log z\,\vec{i} - 2y\vec{j} + \frac{x}{z}\vec{k}$$

$$\boxed{\nabla\phi_{1(1,1,1)} = -2\vec{j} + \vec{k}}$$

$\phi_2 = x^2 y - 2 + z$

$$\frac{\partial\phi_2}{\partial x} = 2xy$$

$$\frac{\partial\phi_2}{\partial y} = x^2$$

$$\frac{\partial\phi_2}{\partial z} = 1$$

$$\nabla\phi_2 = (2xy)\vec{i} + x^2\vec{j} + \vec{k}$$

$$\nabla\phi_2 = 2\vec{i} + \vec{j} + \vec{k}$$

$$\boxed{\nabla\phi_{2(1,1,1)} = 2\vec{i} + \vec{j} + \vec{k}}$$

$|\nabla\phi_1| = \sqrt{4+1} = \sqrt{5}$

$|\nabla\phi_2| = \sqrt{4+1+1} = \sqrt{6}$

$$\cos\theta = \frac{\nabla\phi_1 . \nabla\phi_2}{|\nabla\phi_1| . |\nabla\phi_2|}$$

$$= \frac{(-2\vec{j}+\vec{k}).(2\vec{i}+\vec{j}+\vec{k})}{\sqrt{5}.\sqrt{6}}$$

$$= \frac{-2+1}{\sqrt{30}}$$

$$\cos\theta = \frac{-1}{\sqrt{30}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{-1}{\sqrt{30}}\right)}$$

12) Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-2$ at $(2,-1,2)$.

Sol:. $\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$

13) Find a and b such that the surfaces $ax^2+byz=(a+2)x$ and $4x^2y+z^3=4$ cuts orthogonally at $(1,-1,2)$.

Sol:. Let $\phi_1 = ax^2+byz-(a+2)x$

$\phi_2 = 4x^2y+z^3-4$

$$\nabla\phi_1 = \vec{i}\frac{\partial}{\partial x}(ax^2+byz-(a+2)x) + \vec{j}\frac{\partial}{\partial y}(ax^2+byz-(a+2)x) + \vec{k}\frac{\partial}{\partial z}(ax^2+byz-(a+2)x)$$

$$= \vec{i}(2ax-a-2)+\vec{j}(bz)+\vec{k}(by)$$

$$\nabla\phi_1 = \vec{i}(2ax - a - z) + \vec{j}(bz) + \vec{k}(by)$$

$$\nabla\phi_1(1,-1,2) = \vec{i}(2a(1) - a - 2) + \vec{j}(b(2)) + \vec{k}(b(-1))$$

$$= \vec{i}(2a - a - 2) + \vec{j}(2b) + \vec{k}(-b)$$

$$= \vec{i}(a-2) + \vec{j}(2b) + \vec{k}(-b)$$

$$\boxed{\nabla\phi_1(1,-1,2) = (a-2)\vec{i} + 2b\vec{j} - b\vec{k}}$$

$$\nabla\phi_2 = \vec{i}\frac{\partial}{\partial x}(4x^2y + z^3 - 4) + \vec{j}\frac{\partial}{\partial y}(4x^2y + z^3 - 4) + \vec{k}\frac{\partial}{\partial z}(4x^2y + z^3 - 4)$$

$$= \vec{i}(8xy) + \vec{j}(4x^2) + \vec{k}(3z^2)$$

$$\nabla\phi_2(1,-1,2) = \vec{i}(8(1)(-1)) + \vec{j}(4(1)) + \vec{k}(3(2)^2)$$

$$= -8\vec{i} + 4\vec{j} + 12\vec{k}$$

$$\boxed{\nabla\phi_2(1,-1,2) = -8\vec{i} + 4\vec{j} + 12\vec{k}}$$

Since the surfaces cut orthogonally,

$$\nabla\phi_1 \cdot \nabla\phi_2 = 0$$

$$[(a-2)\vec{i} + 2b\vec{j} - b\vec{k}] \cdot [-8\vec{i} + 4\vec{j} + 12\vec{k}] = 0$$

$$-8a + 16 - 8b + 12b = 0$$

$$-8a + 4b = -16$$

$$\boxed{2a - b = 4} \rightarrow (1)$$

Since the Point $(1,-1,2)$ lies on ϕ,

$$a - 2b - (a+2) = 0 \Rightarrow \boxed{b = -1}$$

Sub $b = -1$ in equ (1) $\Rightarrow \boxed{a = \frac{3}{2}}$