Line Integrals:

Suppose c is an arc and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ is the Position Vector of any Point $P(x,y,z)$ on it and $\vec{f}$ is a Vector Point function at P. Then $\int_c \vec{f}\cdot d\vec{r}$ is called a line integral of $\vec{f}$ over c.

Line integral $\int_A^B \vec{F}\cdot d\vec{r}$ is also known as the total work done by the force $\vec{F}$ during a displacement from A to B.

Problems:

1) Evaluate $\int_c \vec{F}\cdot d\vec{r}$ where $\vec{F} = x^2y^2\vec{i} + y\vec{j}$ and the curve C is $y^2 = 4x$ in the xy-plane from $(0,0)$ to $(4,4)$.

Sol: $\vec{r} = x\vec{i} + y\vec{j}$

$d\vec{r} = dx\vec{i} + dy\vec{j}$

Given: $\vec{F} = x^2y^2\vec{i} + y\vec{j}$

$\vec{F}\cdot d\vec{r} = (x^2y^2\vec{i} + y\vec{j})(dx\vec{i} + dy\vec{j})$

$= x^2y^2dx + ydy$

$$\boxed{\vec{F}\cdot d\vec{r} = x^2y^2dx + ydy}$$

Given $y^2 = 4x$

$2ydy = 4dx$

$ydy = 2dx$

$\therefore \vec{F}\cdot d\vec{r} = x^2y^2dx + 2dx = x^2(4x)dx + 2dx$

$$\int_c \vec{F}\cdot d\vec{r} = \int_0^4 (4x^3 + 2)dx$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^4 (4x^3+2)\,dx$$

$$= \left[\frac{4x^4}{4} + 2x\right]_0^4$$

$$= 4^4 + 8$$

$$= 256 + 8$$

$$= 264$$

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = 264}$$

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2) If $\vec{F} = x^2\vec{i} + xy\vec{j}$ evaluate $\int_C \vec{F}\cdot d\vec{r}$ along the straight line $y = x$ from $(0,0)$ to $(1,1)$.

Sol: $\frac{2}{3}$.

3) If $\vec{F} = 5xy\vec{j} + 2y\vec{j}$, evaluate $\int_C \vec{F}\cdot d\vec{r}$ where C is the part of the curve $y = x^3$ between $x = 1$ and $x = 2$.

Sol:
$$\vec{F} = 5xy\vec{i} + 2y\vec{j}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$\vec{F}\cdot d\vec{r} = 5xy\,dx + 2y\,dy$$

Given: $y = x^3 \Rightarrow dy = 3x^2\,dx$

$$\int_C \vec{F}\cdot d\vec{r} = \int_1^2 5x(x^3)\,dx + 2(x^3)3x^2\,dx$$

$$= \int_1^2 (5x^4 + 6x^5)\,dx$$

$$= \left[\frac{5x^5}{5} + \frac{6x^6}{6}\right]_1^2$$

$$= \left[32 + 64 - (1+1)\right]$$

$$= 94$$

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = 94}$$

4) Find $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = (2y+3)\vec{i} + xz\vec{j} + (yz-x)\vec{k}$ to (2,1,1). along the line joining the Points (0,0,0) to (2,1,1).

Sol:
$$\vec{F} = (2y+3)\vec{i} + xz\vec{j} + (yz-x)\vec{k}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F}\cdot d\vec{r} = (2y+3)dx + xz\,dy + (yz-x)dz$$

The equation of line joining the Points (0,0,0) and (2,1,1) is

$$\frac{x-0}{0-2} = \frac{y-0}{0-1} = \frac{z-0}{0-1}$$

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{1} = t \text{ (say)}$$

$$x = 2t,\quad y = t,\quad z = t$$

$$\vec{F}\cdot d\vec{r} = 2(2t+3)dt + 2t^2 dt + (t^2-2t)dt$$

$$\boxed{\vec{F}\cdot d\vec{r} = (3t^2+2t+6)dt}$$

At $x=0,\ y=0,\ z=0 \Rightarrow t=0$

At $x=2,\ y=1,\ z=1 \Rightarrow t=1$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (3t^2+2t+6)dt = \left[t^3+t^2+6t\right]_0^1 = 8$$

5) If $\vec{F} = (3x^2+6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$ evaluate $\int_c \vec{F}\cdot d\vec{r}$ from (0,0,0) to (1,1,1) along the curve $x=t, y=t^2, z=t^3$.

Sol:
$$\vec{F} = (3x^2+6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$$
$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$
$$\vec{F}\cdot d\vec{r} = (3x^2+6y)dx - 14yz\,dy + 20xz^2dz$$

Given: $x=t,\ y=t^2,\ z=t^3$
$$dx = dt,\ dy = 2t\,dt,\ dz = 3t^2dt$$
$$\vec{F}\cdot d\vec{r} = (3t^2+6t^2)dt - 14(t^2\cdot t^3)2t\,dt + 20(t\cdot t^6)3t^2dt$$
$$= (9t^2 - 28t^6 + 60t^9)dt$$
$$\int_c \vec{F}\cdot d\vec{r} = \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt$$
$$= 5$$
$$\boxed{\int_c \vec{F}\cdot d\vec{r} = 5}$$

Surface Integral:

Let S be a surface whose projection R_{xy} on the xy-plane is such that the points on S have a 1-1 correspondance with the points on R_{xy}. Let ds be a vector element of the area.

Then
$$\iint_S \vec{F}\cdot d\vec{s} = \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{R_{xy}} \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$$

For yz-plane, $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{R_{yz}} \vec{F}\cdot\hat{n}\,\dfrac{dy\,dz}{|\hat{n}\cdot\vec{i}|}$

For xz-plane, $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{R_{xz}} \vec{F}\cdot\hat{n}\,\dfrac{dx\,dz}{|\hat{n}\cdot\vec{j}|}$

The Surface integral $\iint_S \vec{F}\cdot d\vec{s}$ represents the total flux of $\vec{F}$ through the whole surface.