Solution of System of Simultaneous linear first order differential equations with Constant Coefficients.

1) Solve : $\frac{dx}{dt} - y = t$; $\frac{dy}{dt} + x = \sin t$.

The given equation can be written as

$$Dx - y = t \rightarrow (1)$$

$$x + Dy = \sin t \rightarrow (2)$$

$$(1) \times D \Rightarrow D^2x - \cancel{Dy} = D(t) = 1 \rightarrow (3)$$

$$x + \cancel{D}y = \sin t \rightarrow (4)$$

$$\boxed{(D^2+1)x = 1 + \sin t}$$

The auxiliary equ is $m^2 + 1 = 0$

$$m^2 = -1$$

$$m = \pm i$$

Complementary function $C.F = A\cos t + B\sin t$

$$P.I = \frac{1}{D^2+1}(1+\sin t)$$

$$= \frac{1}{D^2+1}(1) + \frac{1}{D^2+1}\sin t \qquad D^2 \rightarrow -1$$

$$= \frac{1}{D^2+1}e^{0t} + \frac{1}{-1+1}\sin t \qquad \frac{1}{D}(\sin t) = -\cos t$$

$$= \frac{1}{D}e^{0t} + \frac{1}{0}\sin t$$

$$= 1 + \frac{t}{2D} \sin t$$

$$= 1 + \frac{t}{2} \frac{1}{D} (\sin t)$$

$$= 1 + \frac{t}{2} (-\cos t)$$

$$P.I = 1 - \frac{t\cos t}{2}$$

$$x = C.F + P.I$$

$$x = A\cos t + B\sin t - \frac{t\cos t}{2} + 1 \longrightarrow (5)$$

$$\frac{dx}{dt} = -A\sin t + B\cos t - \frac{1}{2}(-t\sin t + \cos t)$$

$$= -A\sin t + B\cos t + \frac{t}{2}\sin t - \frac{1}{2}\cos t \longrightarrow (6)$$

Sub (6) in (1), we get

$$y = Dx - t = \frac{dx}{dt} - t$$

$$y = -A\sin t + B\cos t + \frac{t}{2}\sin t - \frac{1}{2}\cos t - t$$

$$x = A\cos t + B\sin t - \frac{t\cos t}{2} + 1$$

2) Solve:. $\frac{dx}{dt} - y = t$, $\frac{dy}{dt} + x = t^2$

Sol:. The given equation can be written as

$$Dx - y = t \longrightarrow (1)$$

$$x + Dy = t^2 \longrightarrow (2)$$

$$(1)\times D \Rightarrow D^2x - Dy = D(t) = 1 \longrightarrow (3)$$

$$x + Dy = t^2 \longrightarrow (4)$$

$$(3)+(4) \Rightarrow (D^2+1)x = t^2+1$$

The auxiliary equ is $m^2+1=0$

$$m^2 = -1$$

$$m = \pm i$$

$$C.F = A\cos t + B\sin t$$

$$P.I = \frac{1}{D^2+1}(t^2+1)$$

$$= (1+D^2)^{-1}(t^2+1)$$

$$= [1 - D^2 + D^4 - \dots](t^2+1)$$

$$= t^2+1-2$$

$$= t^2-1$$

$$x = C.F + P.I$$

$$= A\cos t + B\sin t + t^2 - 1 \longrightarrow (5)$$

$$\frac{dx}{dt} = -A\sin t + B\cos t + 2t \longrightarrow (6)$$

Sub (6) in (1), we get

$$y = \frac{dx}{dt} - t$$

$$y = -A\sin t + B\cos t + 2t - t$$

$$x = A\cos t + B\sin t + t^2 - 1.$$

3) Solve:- $3(1-D)x+4y=3t+1$, $3(D+1)y+2x=e^t$

Sol:-

$$3(1-D)x+4y=3t+1 \rightarrow (1)$$

$$2x+3(D+1)y=e^t \rightarrow (2)$$

$(1)\times 2 \rightarrow \quad 6(1-D)x+8y=6t+2 \rightarrow (3)$

$(2)\times 3(1-D) \Rightarrow 6(1-D)x+9(1-D^2)y=3(1-D)e^t \rightarrow (4)$

$$[8-9(1-D^2)]y=6t+2-\cancel{3e^t}+\cancel{3e^t}$$

$$=6t+2$$

$$(8-9+9D^2)y=6t+2$$

$$(9D^2-1)y=6t+2 \rightarrow (5)$$

The auxiliary equ is $9m^2-1=0$

$$9m^2=1$$

$$m^2=\frac{1}{9}$$

$$m=\pm\frac{1}{3}$$

$$\boxed{C.F = Ae^{\frac{1}{3}t}+Be^{-\frac{1}{3}t}}$$

$$P.I = \frac{1}{9D^2-1}(6t+2)$$

$$=-\frac{1}{1-9D^2}(6t+2)$$

$$=-(1-9D^2)^{-1}(6t+2)$$

$$=-(1+9D^2+\cdots)(6t+2)$$

$$=-(6t+2)$$

$y = C.F + P.I$

$= Ae^{\frac{1}{3}t} + Be^{-\frac{1}{3}t} - (6t+2) \longrightarrow (6)$

$\frac{dy}{dt} = \frac{A}{3}e^{\frac{1}{3}t} - \frac{B}{3}e^{-\frac{1}{3}t} - 6 \longrightarrow (7)$

$2x = e^t - 3\frac{dy}{dt} - 3y \longrightarrow (4)$

$x = \frac{1}{2}\left(e^t - 3\left(\frac{A}{3}e^{\frac{1}{3}t} - \frac{B}{3}e^{-\frac{1}{3}t} - 6\right)\right.$

$\left. - 3\left(Ae^{\frac{1}{3}t} + Be^{-\frac{1}{3}t} - 6t - 2\right)\right]$

$= \frac{1}{2}\left(e^t - Ae^{\frac{1}{3}t} + Be^{-\frac{1}{3}t} + 18 - 3Ae^{\frac{1}{3}t} - 3Be^{-\frac{1}{3}t} + 18t + 6\right]$

$= \frac{1}{2}\left(e^t - 4Ae^{\frac{1}{3}t} - 2Be^{-\frac{1}{3}t} + 18t + 24\right]$

$x = \frac{e^t}{2} - 2Ae^{\frac{1}{3}t} - Be^{-\frac{1}{3}t} + 9t + 12$

$y = Ae^{\frac{1}{3}t} + Be^{-\frac{1}{3}t} - 6t + 2.$