Unit-III

Ordinary Differential Equations

Complex Differentiation

Introduction:-

If x and y are real numbers then $z = x+iy$ is called a Complex number where x is called real Part of z, y is called the imaginary Part of z and the Value of i is $\sqrt{-1}$.

The complex number $x-iy$ is called as the complex Conjugate of z and it is denoted by $\bar{z}$.

ie., $\bar{z} = x-iy$

Note:-

1) $|z| = \sqrt{x^2+y^2}$ 2) $|z^2| = z\bar{z}$ 3) $z\bar{z} = x^2+y^2 = r^2$

4) $|\bar{z}| = |z|$ 5) Real Part of $z = \frac{z+\bar{z}}{2}$

6) Imaginary Part of $z = \frac{z-\bar{z}}{2}$

7) $z = re^{i\theta}$ is called Polar form of z.

8) Amplitude of $z = \theta = \tan^{-1}(y/x)$.

Functions of Complex Variable:-

$w = f(z) = u(x,y) + iv(x,y)$ where $u(x,y)$ and $v(x,y)$ are real Variables.

Single Valued function:

If for each value of z in R there will be only one value of w, then w is called a single valued function of z.

Example: $w = z^2$, $w = \frac{1}{z}$

z:	1	2	-2	3
w:	1	4	4	9

z:	1	2	-2	3
w:	1	$1/2$	$-1/2$	$1/3$

Multiple-Valued function:

If there is more than one value of w corresponding to a given value of z, then w is called a multiple-valued function.

Example: $w = z^{1/2}$

z:	4	9	1
w:	$-2, 2$	$-3, 3$	$1, -1$

Analytic function:

A function $f(z)$ is said to be analytic at a point $z = a$ in a region R if

i) $f(z)$ is differentiable at $z = a$

i) $f(z)$ is differentiable at all points for some neighbourhood of $z = a$.

(or)

A function is said to be analytic at a Point if its derivative exists not only at that Point but also in some neighbourhood of that Point.

Necessary Condition (Cartesian Coordinates) (or) Cauchy-Riemann equations:

If the function $f(z) = u(x,y) + iv(x,y)$ is analytic in a region R of the z plane, then

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ exists

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at every Point in that region.

Sufficient Conditions:

If the function $f(z) = u(x,y) + iv(x,y)$ is analytic in a region R of the z-plane if

(i) u_x, u_y, v_x & v_y are exists and all are continuous,

(ii) $u_x = v_y$ and $u_y = -v_x$.

Necessary Condition (Polar coordinates):

If the function $w = f(z) = u(r,\theta) + iv(r,\theta)$ is analytic in a region R of the z-plane then

i) If $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}$ and $\frac{\partial v}{\partial \theta}$ exists.

(ii) $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

Sufficient Conditions:

If the function $w = f(z) = u(r,\theta) + iv(r,\theta)$ is analytic in a region R of the z-plane, then

(i) $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}$ and $\frac{\partial v}{\partial \theta}$ exists and all are continuous.

Problems:

1) Prove that $w = z^2$ is analytic.

Sol: We know that $z = x + iy$

$$w = z^2 = (x+iy)^2 = x^2 + 2ixy - y^2$$

$$u + iv = (x^2 - y^2) + i(2xy)$$

$u = x^2 - y^2$ $\qquad$ $v = 2xy$

$u_x = 2x$ $\qquad$ $v_x = 2y$

$u_y = -2y$ $\qquad$ $v_y = 2x$

$$u_x = v_y \text{ and } u_y = -v_x$$

It satisfies the Cauchy-Riemann equations.

$w = z^2$ is analytic.

2) Determine whether the function $w = 2xy + i(x^2 - y^2)$ is analytic.

Sol:
$$w = 2xy + i(x^2 - y^2)$$

$$u + iv = 2xy + i(x^2 - y^2)$$