A function is said to be analytic at a point if its derivative exists not only at that point but also in some neighbourhood of that point.

Necessary Condition (Cartesian Coordinates) (or)

Cauchy-Riemann equations:-

If the function $f(z) = u(x,y) + iv(x,y)$ is analytic in a region R of the z plane, then

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ exists

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at every point in that region.

Sufficient Conditions:-

If the function $f(z) = u(x,y) + iv(x,y)$ is analytic in a region R of the z-plane if

(i) u_x, u_y, v_x & v_y are exists and all are continuous.

(ii) $u_x = v_y$ and $u_y = -v_x$.

Necessary Condition (Polar coordinates):-

If the function $w = f(z) = u(r,\theta) + iv(r,\theta)$ is analytic in a region R of the z-plane then

i) If $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}$ and $\frac{\partial v}{\partial \theta}$ exists.

(ii) $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

Sufficient Conditions:

If the function $w = f(z) = u(r,\theta) + iv(r,\theta)$ is analytic in a region R of the z-plane, then

(i) $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}$ and $\frac{\partial v}{\partial \theta}$ exists and all are continuous.

Problems:

1) Prove that $w = z^2$ is analytic.

Sol: We know that $z = x + iy$

$$w = z^2 = (x+iy)^2 = x^2 + 2ixy - y^2$$

$$u + iv = (x^2 - y^2) + i(2xy)$$

$u = x^2 - y^2$ $\quad$ $v = 2xy$

$u_x = 2x$ $\quad$ $v_x = 2y$

$u_y = -2y$ $\quad$ $v_y = 2x$

$$u_x = v_y \text{ and } u_y = -v_x$$

It satisfies the Cauchy-Riemann equations.

$w = z^2$ is analytic.

2) Determine whether the function $w = 2xy + i(x^2 - y^2)$ is analytic.

Sol:
$$w = 2xy + i(x^2 - y^2)$$

$$u + iv = 2xy + i(x^2 - y^2)$$

$u = 2xy \qquad V = x^2 - y^2$

$u_x = 2y \qquad V_x = 2x$

$u_y = 2x \qquad V_y = -2y$

$u_x \neq V_y$ and $u_y \neq -V_x$

It does not Satisfy the Cauchy Riemann equations.

$w = 2xy + i(x^2 - y^2)$ is not analytic.

3) Verify whether $f(z) = \sinh z$ is analytic using CR equations

Sol:

$f(z) = \sinh z$ $\qquad \sin i\theta = i \sinh\theta$, $\cos i\theta = \cosh\theta$

$u + iv = \sinh(x+iy)$

$= \frac{1}{i} \sin i(x+iy)$

$= \frac{1}{i} \sin(ix + i^2 y)$ $\qquad \left[\frac{1}{i} = -i\right]$

$= \frac{1}{i} \sin(ix - y)$

$= \frac{1}{i}\left[\sin ix \cos y - \cos ix \sin y\right]$

$= \frac{1}{i}\left[i \sinh x \cos y - \cosh x \sin y\right]$

$= \sinh x \cos y - \frac{1}{i} \cosh x \sin y$

$= \sinh x \cos y + i \cosh x \sin y$

$u = \sinh x \cos y \qquad u_y = -\sinh x \sin y$

$u_x = \cosh x \cos y$

$v = \cosh x \sin y$

$v_x = \sinh x \sin y$

$v_y = \cosh x \cos y$

$u_x = v_y$ and $u_y = -v_x$

It satisfies the Cauchy Riemann equations

$f(z) = \sinh z$ is analytic.

4) Show that $f(z) = |z|^2$ is nowhere analytic.

Soln: $f(z) = |z|^2$

$u + iv = x^2 + y^2$

$u = x^2 + y^2$; $v = 0$

$u_x = 2x$, $v_x = 0$

$u_y = 2y$, $v_y = 0$

$u_x \neq v_y$ and $u_y \neq -v_x$

It does not satisfies the CR equations.

$\therefore f(z) = |z|^2$ is nowhere analytic.

5) If $u+iv$ is analytic then $v-iu$ is also analytic.

Sol: $u+iv$ is analytic.

ie.,) CR equations are satisfied.

ie.,) $u_x = v_y$ and $u_y = -v_x$

ie.,) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

To Prove: $v - iu$ is also analytic.

we have to Prove $\frac{\partial v}{\partial x} = \frac{\partial(-u)}{\partial y}$ & $\frac{\partial v}{\partial y} = -\frac{\partial(-u)}{\partial x}$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \& \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

we Know that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

Hence, $v - iu$ is also analytic.

6) If $w = e^z$ find $\frac{dw}{dz}$ using complex Variable.

Sol:. $w = e^z$

$$u + iv = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$$
$$= e^x \cos y + i e^x \sin y$$

$u = e^x \cos y \qquad v = e^x \sin y$

$u_x = e^x \cos y \qquad v_x = e^x \sin y$

[Result: If $w = f(z) = u + iv$ then $\frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

finding $\frac{dw}{dz}$ interms of Partial derivatives w.r to x]

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + i e^x \sin y$$
$$= e^x(\cos y + i \sin y)$$
$$= e^x e^{iy} = e^{x+iy} = e^z.$$