Harmonic function:.

An expression of the form $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ is called the Laplace equation in two dimension.

Any function having continuous second order Partial derivatives which Satisfies the Laplace equation is called harmonic function.

Any two harmonic functions u and v such that $f(z) = u+iv$ is analytic are called conjugate harmonic functions.

Note:.

Both real and imaginary Parts of an analytic function are harmonic but the converse need ~~to~~ not to be true.

Problems:.

1) Give an example Such that u and v are harmonic but $u+iv$ is not analytic

Sol:. Let $w = \bar{z}$

$u+iv = x-iy$

$u = x,\ v = -y$

$\frac{\partial u}{\partial x} = 1,\ \frac{\partial u}{\partial y} = 0,\ \frac{\partial v}{\partial x} = 0,\ \frac{\partial v}{\partial y} = -1,\ \frac{\partial^2 u}{\partial x^2} = 0,$

$\frac{\partial^2 u}{\partial y^2} = 0,\ \frac{\partial^2 v}{\partial x^2} = 0,\ \frac{\partial^2 v}{\partial y^2} = 0$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\therefore$ u and v are harmonic.

But $u_x \neq v_y$ and $u_y = -v_x$

$\therefore f(z) = u + iv$ is not analytic.

2) Prove that $u = e^x \cos y$ is a harmonic function.

Sol:. Let $u = e^x \cos y$

$$\frac{\partial u}{\partial x} = e^x \cos y \quad ; \quad \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y \quad ; \quad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\therefore$ u is a harmonic function.

3) Prove that $u = x^2 - y^2$, $v = \frac{-y}{x^2+y^2}$ are harmonic but $u + iv$ is not a regular function.

Sol:. Let $u = x^2 - y^2$

$$\frac{\partial u}{\partial x} = 2x \quad ; \quad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \quad ; \quad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$

$\therefore$ u is a harmonic function.

Let $v = \frac{-y}{x^2+y^2}$

$$\frac{\partial v}{\partial x} = -\frac{\left[(x^2+y^2)(0) - y(2x)\right]}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)^2(2y) - 2xy \cdot 2(x^2+y^2)(2x)}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)^2(2y) - 8x^2y(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2) - 8x^2y}{(x^2+y^2)^3}$$

$$= \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = -\frac{\left[(x^2+y^2) - y \cdot 2y\right]}{(x^2+y^2)^2} = \frac{-(x^2-y^2)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)^2 2y - (y^2-x^2)\, 2(x^2+y^2)\, 2y}{(x^2+y^2)^4}$$

$u_x = v_y$

$-v_x = u_y$

$$= \frac{(x^2+y^2)^2(2y) - 4y(y^2-x^2)(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)(2y) - 4y(y^2-x^2)}{(x^2+y^2)^3}$$

$$= \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{2y^3 - 6x^2y}{(x^2+y^2)^3} + \frac{6x^2y - 2y^3}{(x^2+y^2)^3} = 0$$

$\therefore$ V is a harmonic function

But $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

$f(z) = u + iv$ is not analytic (or) not regular function.

Construction of Conjugate harmonic function :.

Method : 1

Suppose u is given, then

$$v = \int \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) + c,$$ where c is a constant.

Method : 2

Suppose v is given, then

$$u = \int \left(\frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy \right) + c,$$ where c is a constant.