$$= \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{2y^3 - 6x^2y}{(x^2+y^2)^3} + \frac{6x^2y - 2y^3}{(x^2+y^2)^3} = 0$$

$\therefore$ v is a harmonic function

But $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

$f(z) = u + iv$ is not analytic (or) not regular function.

Construction of Conjugate harmonic function :.

Method :1

Suppose u is given, then

$$v = \int \left(-\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy \right) + c,$$ where c is a

constant.

Method : 2

Suppose v is given, then

$$u = \int \left(\frac{\partial v}{\partial y}\, dx - \frac{\partial v}{\partial x}\, dy \right) + c,$$ where c is a

Constant.

Problems:

1) Show that the function $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic and find its harmonic conjugate.

Sol: Let $u = \frac{1}{2}\log(x^2+y^2)$

$$u_x = \frac{1}{2}\frac{1}{x^2+y^2}2x = \frac{x}{x^2+y^2}$$

$$u_{xx} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$u_y = \frac{1}{2}\cdot\frac{1}{x^2+y^2}2y = \frac{y}{x^2+y^2}$$

$$u_{yy} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$\therefore u_{xx} + u_{yy} = 0$

u satisfies the Laplace equation.

u is harmonic.

$$v = \int\left(-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy\right) + c$$

$$= \int\left(\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right) + c$$

$$= \int\frac{x\,dy - y\,dx}{x^2+y^2} + c$$

$$= \int \frac{x\,dy - y\,dx}{x^2(1+y^2/x^2)}$$

$$= \int \frac{d(y/x)}{1+y^2/x^2} + c$$

$$V = \tan^{-1}(y/x) + c$$

2) Prove that the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic find the conjugate harmonic function.

Sol: Let $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$u_x = 3x^2 - 3y^2 + 6x$ $\qquad u_{xx} = 6x + 6$

$u_y = -6xy - 6y$ $\qquad u_{yy} = -6x - 6 = -6(x+1)$

$\therefore u_{xx} + u_{yy} = 0$

$\therefore u$ satisfies the Laplace equation.

$\therefore u$ is harmonic

Now, $$v = \int \left(-\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy\right) + c$$

$$= \int (6xy + 6y)\,dx + (3x^2 - 3y^2 + 6x)\,dy + c$$

$$= \frac{6x^2y}{2} + 6xy + 3x^2y - \frac{3y^3}{3} + 6xy + c$$

$$= \frac{1}{2}\left[6x^2y + 12xy + 6x^2y - 2y^3 + 12xy + 2c\right]$$

$$V = 6x^2y + 12xy - y^3 + c.$$

3) Show that $u = \cos x \cosh y$ is harmonic and hence find its harmonic conjugate.

Sol:- $u = \cos x \cosh y$

$u_x = -\sin x \cosh y$ $\quad$ $u_y = \cos x \sinh y$

$u_{xx} = -\cos x \cosh y$ $\quad$ $u_{yy} = \cos x \cosh y$

$$u_{xx} + u_{yy} = 0$$

$\therefore$ u satisfies the Laplace equation.

$\therefore$ u is harmonic.

Now, $v = \int \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy\right) + c$

$= \int (-\cos x \sinh y\, dx) + (-\sin x \cosh y) dy + c$

$= -\sin x \sinh y - \sin x \sinh y + c$

$v = -2\sin x \sinh y + c.$

Conjugate Harmonic :-

If $f(z) = u + iv$ is an analytic function of z then u and v are harmonic functions and also v is the harmonic conjugate of u and u is the harmonic conjugate of v.

To find harmonic conjugate : If u is given (Real part)

$$v = \int \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy\right) + c.$$

Here, $\frac{\partial u}{\partial x}$ is the terms not involving 'x'.

$$v = \int \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy\right) + c.$$

If v is given (imaginary Part)

$$u = \int -\frac{\partial v}{\partial y} dx + \frac{\partial v}{\partial x} dy + c.$$

1) Verify $u = 2x - x^3 + 3xy^2$ is harmonic and determine its harmonic conjugate.

Sol: Given $u = 2x - x^3 + 3xy^2$

Condition is $u_{xx} + u_{yy} = 0$

$u_x = 2 - 3x^2 + 3y^2$ $\qquad$ $u_y = 6xy$

$u_{xx} = -6x$ $\qquad$ $u_{yy} = 6x$

$$u_{xx} + u_{yy} = -6x + 6x = 0$$

$\therefore$ u is harmonic function.

Since u is given harmonic conjugate

$$v = \int \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy\right) + c$$

$\frac{\partial u}{\partial x}$ is the term not involving 'x'.

$$v = \int \left(-6xy\, dx + (2 + 3y^2)\, dy\right) + c$$

$$= -6y \frac{x^2}{2} + 2y + \frac{3y^3}{3} + c$$

$$v = -3yx^2 + 2y + y^3 + c.$$

2) Prove that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic and determine its conjugate harmonic.

Sol:: $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

Condition is $U_{xx} + U_{yy} = 0$

$U_x = 3x^2 - 3y^2 + 6x$ $\quad$ $U_y = -6xy - 6y$

$U_{xx} = 6x + 6$ $\quad$ $U_{yy} = -6x - 6$

$U_{xx} + U_{yy} = 6x + 6 - 6x - 6$

$U_{xx} + U_{yy} = 0$

Therefore u is a harmonic function.

Since u is given harmonic conjugate v is

$$V = \int \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) + c$$

Here $\frac{\partial u}{\partial x}$ is the term not involving x,

$$v = \int (-(-6xy) - 6y)\, dx + (-3y^2)\, dy + c$$

$$= \int 6y \frac{x^2}{2} + 6xy - \frac{3y^3}{3} + c$$

$$v = 3x^2 y + 6xy - y^3 + c.$$