Construction of Analytic functions:

Milne Thomson method:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad \text{by C.R.}$$

$$\phi_1(z,0) = \left(\frac{\partial u}{\partial x}\right)_{(z,0)} \qquad \phi_2(z,0) = \left(\frac{\partial u}{\partial y}\right)_{(z,0)}$$

$$f'(z) = \phi_1(z,0) - i\phi_2(z,0)$$

$$\int f'(z)\,dz = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$$

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + c$$

where c is a complex constant.

To find f(z) when v is given:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$\phi_1(z,0) = \left(\frac{\partial v}{\partial y}\right)_{(z,0)} \qquad \phi_2(z,0) = \left(\frac{\partial v}{\partial x}\right)_{(z,0)}$$

$$f'(z) = \phi_1(z,0) + i\phi_2(z,0)$$

$$\int f'(z)\,dz = \int \phi_1(z,0)\,dz + i\int \phi_2(z,0)\,dz$$

$$f(z) = \int \phi_1(z,0)\,dz + i\int \phi_2(z,0)\,dz + c.$$

Problems:

1) Determine the analytic function whose real part is $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$.

Sol: Given $u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_1(z,0) = \frac{(1-\cos 2z)(2\cos 2z) - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{(1-\cos 2z)(2\cos 2z) - 2(1-\cos^2 2z)}{(1-\cos 2z)^2}$$

$$= \frac{(1-\cos 2z)(2\cos 2z) - 2(1-\cos 2z)(1+\cos 2z)}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2(1+\cos 2z)}{(1-\cos 2z)} = \frac{2\cos 2z - 2 - 2\cos 2z}{1-\cos 2z}$$

$$= \frac{-2}{1-\cos 2z} = -\frac{1}{\left(\frac{1-\cos 2z}{2}\right)}$$

$$= \frac{-1}{\sin^2 z} = -\operatorname{cosec}^2 z$$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x\,(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_2(z,0) = 0$$

By Milne's Thomson method,

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$$

$$= \int -\operatorname{cosec}^2 z\,dz - 0$$

$$\boxed{f(z) = \cot z + c}$$

2) Find the analytic functions $f(z) = u+iv$ given that

i) $-2v = e^x(\cos y - \sin y)$ ii) $2u+v = e^x(\cos y - \sin y)$

iii) $u-2v = e^x(\cos y - \sin y)$

Sol:- i) Let $f(z) = u+iv \rightarrow (1)$

$-if(z) = -iu+v \rightarrow (2)$

$-2\times(2) \Rightarrow 2if(z) = 2iu - 2v$

$2if(z) = (-2v) + i(2u)$

$F(z) = U+iV$

Where $F(z) = 2if(z)$

$U = -2v$, $V = 2u$

$\phi_1(x,y) = \frac{\partial U}{\partial x} = e^x(\cos y - \sin y)$

$\phi_1(z,0) = e^z$

$\phi_2(x,y) = \frac{\partial U}{\partial y} = e^x(-\sin y - \cos y)$

$= e^z(-1)$

$\phi_2(z,0) = -e^z$

By Milne's thomson method,

$F(z) = \int \phi_1(z,0)dz - i\int \phi_2(z,0)\,dz$

$= \int e^{z}dz - i\int -e^z dz$

$= \int e^z dz + i\int e^z dz$

$= (1+i)\int e^z dz$

$2if(z) = (1+i)e^z + c_1$

$f(z) = \frac{1+i}{2i}e^z + c$

$= \frac{1-i}{2}e^z + c.$

ii) $2u+v = e^x(\cos y - \sin y)$

$(1)\times 2 \Rightarrow 2u + 2iv = 2f(z) \rightarrow (3)$

$\div 2 \Rightarrow v - iu = -if(z) \rightarrow (4)$

$(3)+(4) \Rightarrow (2u+v)+i(2v-u) = (2-i)f(z)$

$F(z) = U + iV \quad , \quad F(z) = (2-i)f(z)$

$U = 2u+v, \quad V = 2v-u$

$U = e^x(\cos y - \sin y)$

$\phi_1(x,y) = \frac{\partial U}{\partial x} = e^x(\cos y - \sin y)$

$\phi_1(z,0) = e^z$

$\phi_2(x,y) = \frac{\partial U}{\partial y} = e^x(-\sin y - \cos y)$

$\phi_2(z,0) = -e^z$

By Milne's thomsan method,

$F(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$

$= \int e^z dz - i\int -e^z dz$

$= \int e^z dz + i\int e^z dz$

$= (1+i)\int e^z dz = (1+i)e^z + c_1$

$(2-i)f(z) = (1+i)e^z + c_1$

$f(z) = \frac{1+i}{2-i}e^z + c$

$= \frac{1+3i}{5}e^z + c$

iii) $u - 2v = e^x(\cos y - \sin y)$

① $- 2 \times$ ② we get

$$(u-2v) + i(2u+v) = (1+2i) f(z)$$

$$F(z) = U + iV$$

$$F(z) = (1+2i) f(z)$$

$$U = u - 2v, \quad V = 2u+v$$

$$U = u - 2v = e^x(\cos y - \sin y)$$

$$\phi_1(x,y) = \frac{\partial U}{\partial x} = e^x(\cos y - \sin y)$$

$$\phi_1(z,0) = e^z$$

$$\phi_2(x,y) = \frac{\partial U}{\partial y} = e^x(-\sin y - \cos y)$$

$$\phi_2(z,0) = -e^z$$

By Milne's thomson method

$$F(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$$

$$= \int e^z dz - i\int -e^z dz$$

$$= \int e^z dz + i\int e^z dz$$

$$= (1+i)\int e^z dz$$

$$(1+2i) f(z) = (1+i)e^z + c_1$$

$$f(z) = \frac{1+i}{1+2i} e^z + c$$

$$f(z) = \frac{3-i}{5} e^z + c$$