



Construction of Analytic functions..

Milne Thomson method:

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \quad \text{by C.R.}$$

$$\phi_1(z,0) = \left(\frac{\partial u}{\partial x} \right)_{(z,0)} \quad \phi_2(z,0) = \left(\frac{\partial u}{\partial y} \right)_{(z,0)}$$

$$f'(z) = \phi_1(z,0) - i \phi_2(z,0)$$

$$\int f'(z) dz = \int \phi_1(z,0) dz - i \int \phi_2(z,0) dz$$

$$f(z) = \int \phi_1(z,0) dz - i \int \phi_2(z,0) dz + c$$

where c is a complex constant.

To find $f(z)$ when v is given:

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x}$$

$$\phi_1(z,0) = \left(\frac{\partial v}{\partial y} \right)_{(z,0)} \quad \phi_2(z,0) = \left(\frac{\partial v}{\partial x} \right)_{(z,0)}$$

$$f'(z) = \phi_1(z,0) + i \phi_2(z,0)$$

$$\int f'(z) dz = \int \phi_1(z,0) dz + i \int \phi_2(z,0) dz$$

$$f(z) = \int \phi_1(z,0) dz + i \int \phi_2(z,0) dz + c.$$

Problems:

(1) Determine the analytic function whose real part is

$$\frac{\sin 2x}{\cosh 2y - \cos 2x}.$$

Sol: Given $u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$



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$$\phi_1(x, y) = \frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x)(2 \cos 2x) - \sin 2x (2 \sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_1(z, 0) = \frac{(1 - \cos 2z)(2 \cos 2z) - 2 \sin^2 2z}{(1 - \cos 2z)^2}$$

$$= \frac{(1 - \cos 2z)(2 \cos 2z) - 2(1 - \cos^2 2z)}{(1 - \cos 2z)^2}$$

$$= \frac{(1 - \cos 2z)(2 \cos 2z) - 2(1 - \cos 2z)(1 + \cos 2z)}{(1 - \cos 2z)^2}$$

$$= \frac{2 \cos 2z - 2(1 + \cos 2z)}{(1 - \cos 2z)} = \frac{2 \cos 2z - 2 - 2 \cos 2z}{1 - \cos 2z}$$

$$= \frac{-2}{1 - \cos 2z} = -\frac{1}{\left(\frac{1 - \cos 2z}{2}\right)}$$

$$= \frac{-1}{\sin^2 z} = -\operatorname{cosec}^2 z$$

$$\phi_2(x, y) = \frac{\partial u}{\partial y} = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x (2 \sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{-2 \sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$$\phi_2(z, 0) = 0$$

By Milne's Thomson method,

$$f(z) = \int \phi_1(z, 0) dz - i \int \phi_2(z, 0) dz$$

$$= \int -\operatorname{cosec}^2 z dz - 0$$

$$\boxed{f(z) = \cot z + C}$$



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2) Find the analytic functions $f(z) = u + iv$ given that

i) $-2v = e^x(\cos y - \sin y)$ ii) $2u + v = e^x(\cos y - \sin y)$

iii) $u - 2v = e^x(\cos y - \sin y)$

Sol: i) Let $f(z) = u + iv \rightarrow (1)$

$-if(z) = -iu + v \rightarrow (2)$

$-2 \times (2) \Rightarrow 2if(z) = 2iu - 2v$

$2if(z) = (-2v) + i(2u)$

$F(z) = u + iv$

Where $F(z) = 2if(z)$

$u = -2v, v = 2u$

$\phi_1(x, y) = \frac{\partial u}{\partial x} = e^x(\cos y - \sin y)$

$\phi_1(z, 0) = e^z$

$\phi_2(x, y) = \frac{\partial u}{\partial y} = e^x(-\sin y - \cos y)$
 $= e^z(-1)$
 $\phi_2(z, 0) = -e^z$

By Milne's Thomson method,

$F(z) = \int \phi_1(z, 0) dz - i \int \phi_2(z, 0) dz$

$= \int e^z dz - i \int -e^z dz$

$= \int e^z dz + i \int e^z dz$

$= (1+i) \int e^z dz$

$2if(z) = (1+i)e^z + c_1$

$f(z) = \frac{1+i}{2i} e^z + c$

$= \frac{1-i}{2} e^z + c$



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$$ii) 2u + v = e^x (\cos y - \sin y)$$

$$(1) \times 2 \Rightarrow 2u + 2iv = 2f(z) \rightarrow (3)$$

$$+ 2 \Rightarrow v - iu = -if(z) \rightarrow (4)$$

$$(3) + (4) \Rightarrow (2u + v) + i(2v - u) = (2 - i)f(z)$$

$$F(z) = u + iv, \quad F(z) = (2 - i)f(z)$$

$$u = 2u + v, \quad v = 2v - u$$

$$u = e^x (\cos y - \sin y)$$

$$\phi_1(x, y) = \frac{\partial u}{\partial x} = e^x (\cos y - \sin y)$$

$$\phi_1(z, 0) = e^z$$

$$\phi_2(x, y) = \frac{\partial v}{\partial y} = e^x (-\sin y - \cos y)$$

$$\phi_2(z, 0) = -e^z$$

By Milne's thomsan method,

$$F(z) = \int \phi_1(z, 0) dz - i \int \phi_2(z, 0) dz$$

$$= \int e^z dz - i \int -e^z dz$$

$$= \int e^z dz + i \int e^z dz$$

$$= (1+i) \int e^z dz = (1+i)e^z + C_1$$

$$(2-i)f(z) = (1+i)e^z + C_1$$

$$f(z) = \frac{1+i}{2-i} e^z + C$$

$$= \frac{(1+3i)}{5} e^z + C$$



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$$\text{iii) } u - 2v = e^x (\cos y - \sin y)$$

① - 2 × ② we get

$$(u - 2v) + i(2u + v) = (1 + 2i)f(z)$$

$$F(z) = u + iv$$

$$F(z) = (1 + 2i)f(z)$$

$$u = u - 2v, \quad v = 2u + v$$

$$v = u - 2v = e^x (\cos y - \sin y)$$

$$\phi_1(x, y) = \frac{\partial u}{\partial x} = e^x (\cos y - \sin y)$$

$$\phi_1(z, 0) = e^z$$

$$\phi_2(x, y) = \frac{\partial v}{\partial y} = e^x (-\sin y - \cos y)$$

$$\phi_2(z, 0) = -e^z$$

By Milne's Thomson method

$$F(z) = \int \phi_1(z, 0) dz - i \int \phi_2(z, 0) dz$$

$$= \int e^z dz - i \int -e^z dz$$

$$= \int e^z dz + i \int e^z dz$$

$$= (1 + i) \int e^z dz$$

$$(1 + 2i)f(z) = (1 + i)e^z + C_1$$

$$f(z) = \frac{1+i}{1+2i} e^z + C$$

$$f(z) = \frac{3-i}{5} e^z + C$$